3. Second-order logic

Fix some vocabulary τ:

alphabet: $\Sigma_2(\tau) :=_{def} \Sigma_2 \cup \tau$:

- vocabulary τ
- countable set of object variables $x_0, x_1, x_2, \dots$
- countable set of relation variables $X_0, X_1, X_2, \dots$ (with arities)
- equation symbol $=$
- connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- quantifier symbols $\exists, \forall$
- parentheses $(,)$

$\Big\}\ \Sigma_2$

Definition 1.

The set $T_2(\tau)$ of $SO(\tau)$-terms is defined as follows:

- atomic terms:

  $c \in \tau$ constant symbol $\rightarrow$ $c \in T_2(\tau)$

  $x_i \in \tau$ object variable $\rightarrow$ $x_i \in T_2(\tau)$

- composite terms:

  $t_1, \dots, t_n \in T_2(\tau)$, f n-ary function symbol

  $\Rightarrow f(t_1, \dots, t_n) \in T_2(\tau)$

  (or: $ft_1 \dots t_n \in T_2(\tau)$)

Definition 2.

The set $SO(\tau)$ of SO(τ)-formulas is defined as follows:

- atomic formulas:

  $t_1, t_2 \in T_2(\tau) \Rightarrow (t_1 = t_2) \in SO(\tau)$

  $t_1, \dots, t_n \in T_2(\tau)$, R n-ary relation symbol
  $\Rightarrow R(t_1, \dots, t_n) \in SO(\tau)$

  $\rightarrow$ $t_1, \dots, t_n \in T_2(\tau)$, X_i n-ary relation variable
  $\rightarrow X_i(t_1, \dots, t_n) \in SO(\tau)$

- composite formulas:

  $\varphi \in SO(\tau) \Rightarrow \neg\varphi \in SO(\tau)$

  $\varphi, \psi \in SO(\tau) \rightarrow (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi) \in SO(\tau)$

  $\varphi \in SO(\tau)$, x_i object variable $\Rightarrow \exists x_i \varphi, \forall x_i \varphi \in SO(\tau)$

  $\rightarrow$ $\varphi \in SO(\tau)$, X_i relation variable $\rightarrow \exists X_i \varphi, \forall X_i \varphi \in SO(\tau)$

We define semantics of second-order logic:

Definition 3.

Let τ be a vocabulary.
A pair $I = (\mathfrak{A}, \beta)$ is called τ-interpretation iff

(1.) $\mathfrak{A} = (A, \alpha)$ is a τ-structure

(2.) x_i object variable $\rightarrow \beta(x_i) \in A$

(3.) X_i n-ary relation variable
$\rightarrow \beta(X_i) : A^n \rightarrow \{0,1\}$

Definition 4.

Let τ be a vocabulary. Let $\mathcal{I} = (\mathfrak{A}, \beta)$ be a τ-interpretation, $\mathfrak{A} = (A, \alpha)$ τ-structure.

(1.) The interpretation $[\![t]\!]^{\mathcal{I}}$ of a τ-term t is defined as follows:

- $[\![c]\!]^{\mathcal{I}} =_{def} \alpha(c)$ for all const. symbols $c \in T_2(\tau)$
- $[\![x_i]\!]^{\mathcal{I}} =_{def} \beta(x_i)$ for all object var. $x_i \in T_2(\tau)$
- $[\![f t_1 \dots t_n]\!]^{\mathcal{I}} =_{def} \alpha(f)([\![t_1]\!]^{\mathcal{I}}, \dots, [\![t_n]\!]^{\mathcal{I}})$ for all n-ary function symbol $f \in \tau$

(2.) The interpretation $[\![\varphi]\!]^{\mathcal{I}}$ of a formula $\varphi \in SO(\tau)$ is defined as follows:

- atomic formulas:

$$[\![t_1 = t_2]\!]^{\mathcal{I}} =_{def} \begin{cases} 1 & \text{if } [\![t_1]\!]^{\mathcal{I}} = [\![t_2]\!]^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$$

$$[\![R t_1 \dots t_n]\!]^{\mathcal{I}} =_{def} \begin{cases} 1 & \text{if } ([\![t_1]\!]^{\mathcal{I}}, \dots, [\![t_n]\!]^{\mathcal{I}}) \in \alpha(R) \\ 0 & \text{otherwise} \end{cases}$$

$$[\![X_i t_1 \dots t_n]\!]^{\mathcal{I}} =_{def} \begin{cases} 1 & ([\![t_1]\!]^{\mathcal{I}}, \dots, [\![t_n]\!]^{\mathcal{I}}) \in \beta(X_i) \\ 0 & \text{otherwise} \end{cases}$$

- composite formulas:

$[\![\neg\varphi]\!]^{\mathcal{I}} =_{def} \text{non}([\![\varphi]\!]^{\mathcal{I}})$

$[\![\varphi \wedge \psi]\!]^{\mathcal{I}} =_{def} \text{et}([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$

$[\![\varphi \vee \psi]\!]^{\mathcal{I}} =_{def} \text{vel}([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$

$[\![\varphi \to \psi]\!]^{\mathcal{I}} =_{def} \text{seq}([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$

$$[\![\varphi \Leftrightarrow \psi]\!]^{\mathcal{I}} =_{def} aeq([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$$

$$[\![\exists x_i \varphi]\!]^{\mathcal{I}} =_{def} \max_{\mathcal{I}' \overset{x_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$

$$[\![\forall x_i \varphi]\!]^{\mathcal{I}} =_{def} \min_{\mathcal{I}' \overset{x_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$

$\rightarrow$ $$[\![\exists X_i \varphi]\!]^{\mathcal{I}} =_{def} \max_{\mathcal{I}' \overset{X_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$

$\rightarrow$ $$[\![\forall X_i \varphi]\!]^{\mathcal{I}} =_{def} \min_{\mathcal{I}' \overset{X_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$

<u>Example:</u> $\tau = \{0, s\}$, s unary function symbol, 0 const. symbol. Consider

$$\varphi =_{def} \forall X\,(((X(0) \wedge \forall x\,(X(x) \rightarrow X(s(x)))) \rightarrow \forall y\, X(y))$$

Consider $\mathcal{I} = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (\mathbb{N}, \alpha)$:

- $\alpha(0) =_{def} 0$
- $\alpha(s) =_{def} n \mapsto n+1$
- β irrelevant

We obtain:

$$[\![\varphi]\!]^{\mathcal{I}} = \min_{\mathcal{I}' \overset{X}{=} \mathcal{I}} seq([\![X(0) \wedge \forall x(X(x) \rightarrow X(s(x)))]\!]^{\mathcal{I}'}, [\![\forall y X(y)]\!]^{\mathcal{I}'})$$

$$= \min_{\mathcal{I}' \overset{X}{=} \mathcal{I}} seq(et([\![X(0)]\!]^{\mathcal{I}'}, [\![\forall x(X(x) \rightarrow X(s(x)))]\!]^{\mathcal{I}'}), [\![\forall y X(y)]\!]^{\mathcal{I}'})$$

Suppose $[\![\forall y\, X(y)]\!]^{\mathcal{I}'} = 0$. We have:

$$0 = [\![\forall y\, X(y)]\!]^{\mathcal{I}'}$$

$$= \min_{\mathcal{I}'' \overset{y}{=} \mathcal{I}'} [\![X(y)]\!]^{\mathcal{I}''}$$

$$= \min_{I'' \overset{x}{=} I'} \beta'(X)(\llbracket y \rrbracket^{I''})$$

$$= \min_{n \in \mathbb{N}} \beta'(X)(n)$$

Let n be minimal subject to $n \notin \beta'(X)$, i.e., $\beta'(X)(n) = 0$.

Two cases:

(i) $n = 0$: $\llbracket X(0) \rrbracket^{I'} = \beta'(X)(\alpha(0)) = \beta'(X)(0) = 0$

(ii) $n > 0$: $\llbracket \forall x (X(x) \to X(s(x))) \rrbracket^{I'}$

$$= \min_{I'' \overset{x}{=} I'} seq(\llbracket X(x) \rrbracket^{I''}, \llbracket X(s(x)) \rrbracket^{I''})$$

$$= \min_{I'' \overset{x}{=} I'} seq(\beta'(X)(\llbracket x \rrbracket^{I''}), \beta'(X)(\underbrace{\llbracket x \rrbracket^{I''} + 1}_{\alpha(s)(\llbracket x \rrbracket^{I''})}))$$

$$= \min_{m \in \mathbb{N}} seq(\beta'(X)(m), \beta'(X)(m+1))$$

$$= 0 \qquad (\text{for } m = n-1)$$

Hence, $\llbracket \varphi \rrbracket^{I} = 1$. Induction principle is correct on $\mathbb{N}$.

Definition 5.

Let τ be a vocabulary. Let I be a τ-interpretation.

(1.) I is said to be a model of $\varphi \in SO(\tau)$ iff $\llbracket \varphi \rrbracket^{I} = 1$.

(2.) I is said to be a model of $\Phi \subseteq SO(\tau)$ iff $\llbracket \varphi \rrbracket^{I} = 1$ for all $\varphi \in \Phi$ (i.e., $\llbracket \Phi \rrbracket^{I} = 1$)

(3.) $\Phi \models \varphi$ iff $\llbracket \Phi \rrbracket^{I} \leq \llbracket \varphi \rrbracket^{I}$ for all I.

Example:

(1.) $\Phi =_{def} \{\varphi_1, \varphi_2, \varphi_3\}$ where

$\varphi_1 =_{def} \quad \forall x \;\; \neg s(x) = 0$

$\varphi_2 =_{def} \quad \forall x \forall y \;\; (s(x) = s(y) \rightarrow x = y)$

$\varphi_3 =_{def} \quad \varphi$

Then, $(\mathbb{N}, s, 0)$ s.t. $s(x) = x+1$ is a model of Φ

Dedekind: Each model of Φ is isomorphic to $(\mathbb{N}, s, 0)$

(2.) $\varphi_f(X) =_{def} \quad \forall x \exists y \; X(x,y) \wedge$
$\forall x \forall y \forall z \; ((X(x,y) \wedge X(x,z)) \rightarrow y = z)$

X is a total function

$\varphi_{1\text{-}1}(X) =_{def} \quad \varphi_f(X) \wedge \forall x \forall y \forall z ((X(x,z) \wedge X(y,z)) \rightarrow x = y)$

X is injective

$\varphi_{fin} =_{def} \forall X \, (\varphi_{1\text{-}1}(X) \rightarrow \forall y \exists x \; X(x,y))$

Then,

$\mathfrak{A} = (A, \alpha)$ model of φ_{fin}

$\Leftrightarrow$ ~~ea~~ each injective function $f: A \rightarrow A$ is onto (surjective)

$\Leftrightarrow$ A is finite

(3.) $\forall x \forall y \, ((x = y) \leftrightarrow \forall X (X(x) \leftrightarrow X(y)))$ is true

(4.) $\forall X \forall x \forall y \, ((X(x) \wedge X(y)) \rightarrow \neg x = y)$ SO

$\forall x \forall y \;\; \neg x = y$ FO

Theorem 6.

The compactness theorem for $\models$ does not hold for $SO(\tau)$.
That is, there ex. $\Phi \subseteq SO(\tau)$ and $\varphi \in SO(\tau)$ s.t. the following equivalence is <u>not</u> true:

$$\Phi \models \varphi \iff \text{there is a finite } \Phi_0 \subseteq \Phi \text{ s.t. } \Phi_0 \models \varphi$$

Proof: We show ex. of Φ, φ s.t. $\Phi \models \varphi$ and $\Phi_0 \not\models \varphi$ for all finite $\Phi_0 \subseteq \Phi$. Consider the following formulas for $k \geq 2$:

$$\varphi_k^{\geq} =_{def} \exists x_1 \ldots \exists x_k \bigwedge_{i=1}^{k-1} \bigwedge_{j=i+1}^{k} \neg x_i = x_j$$

That is: (A, α) is a model of $\varphi_k^{\geq} \iff \|A\| \geq k$

Define

$$\Phi =_{def} \{\varphi_k^{\geq} \mid k \geq 2\} \cup \{\varphi_{fin}\}$$

Φ has no model; thus, $\Phi^{\models} = SO(\tau)$. Choose $\varphi \in SO(\tau)$.
Hence, $\Phi \models \varphi \wedge \neg\varphi$.

Assume $\Phi_0 \models \varphi \wedge \neg \varphi$ for some finite $\Phi_0 \subseteq \Phi$.
Then, Φ_0 has a model: Let k be maximal subject to $\varphi_k^{\geq} \in \Phi_0$. Each ~~set~~ finite set A s.t. $\|A\| \geq k$ is a model of Φ_0. However, for each model I of Φ_0, $[\![\varphi \wedge \neg\varphi]\!]^I = 1$, i.e., $[\![\varphi]\!]^I = [\![\neg\varphi]\!]^I = 1$. $\lightning$

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Theorem 7.

$SO(\tau)$ is incomplete for each vocabulary τ, i.e., there ex. no set of derivation rules which is correct and complete.

Proof: Suppose there is a set of derivation rules such that $\vdash$ is defined which is correct, i.e., $\Phi^{\vdash} \subseteq \Phi^{\models}$, and complete, i.e., $\Phi^{\models} \subseteq \Phi^{\vdash}$. Thus, $\Phi^{\models} = \Phi^{\vdash}$ for all sets $\Phi \subseteq SO(\tau)$

However, applying a derivation rules connects a finite number of formulas to obtain a new formula. Thus, compactness theorem always holds for $\vdash$. Since $\Phi^{\models} = \Phi^{\vdash}$, by assumption, the compactness theorem holds for $\models$. $\lightning$ ∎

4. Finite model theory

want to prove that $(aa)^+$ is not FO-definable

4.1 Predicate logic on words

Words as structures:

- Σ finite, non-empty alphabet
- vocabulary $\tau_\Sigma =_{def} \{<\} \cup \{P_a \mid a \in \Sigma\}$ where
  $<$ is a binary relation symbol
  P_a is a unary relation symbol