5. Alternation

Alternation is another computational mode, like nondeterminism

5.1 Alternating Turing machines

An alternating Turing machine is defined to be an NTM with foll. types of states:

- accepting halting state
- rejecting halting state
- existential states
- universal states

} nondeterminism

Accordingly, configurations (tape inscr., head pos., state) are classified into

accepting, rejecting, existential, universal

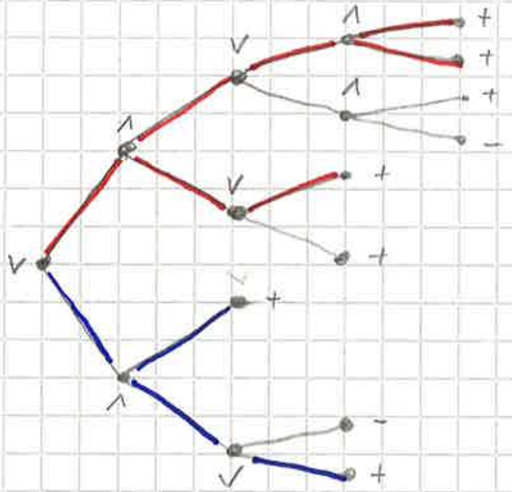
depending on the state.

Let $\beta_M(x)$ be comp. tree of NTM M on input x, i.e., vertices stand for configurations, children are successor configurations.

Define an accepting sub-tree β of $\beta_M(x)$:

- β contains root of $\beta_M(x)$ (initial conf.)
- for each existential conf. in β, β contains exactly one child of $\beta_M(x)$
- for each universal conf. in β, β contains all children of $\beta_M(x)$
- leaves of β are accepting configurations

Example:



accepting subtrees

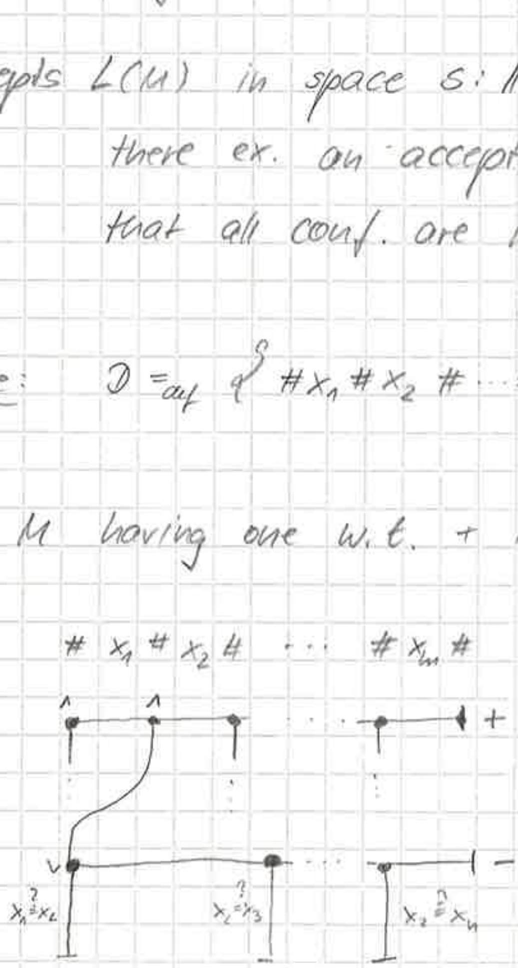
An ATM M

- accepts x $\Leftrightarrow_{def}$ there ex. an accepting subtree of $\beta_M(x)$
- accepts a set $L(M) =_{def} \{x \mid M \text{ accepts } x\}$
- accepts $L(M)$ in time $t: \mathbb{N} \to \mathbb{N}$ $\Leftrightarrow_{def}$ for each $x \in L(M)$, there ex. an accepting subtree of $\beta_M(x)$ of height $\leq t(|x|)$
- accepts $L(M)$ in space $s: \mathbb{N} \to \mathbb{N}$ $\Leftrightarrow_{def}$ for each $x \in L(M)$, there ex. an accepting subtree of $\beta_M(x)$ such that all conf. are length-bounded by $s(|x|)$

Example: $D =_{def} \{ \#x_1\#x_2\#\cdots\#x_m\# \mid x_1,\dots,x_m \in \{0,1\}^* \wedge (\forall i)(\exists j)[i \neq j \wedge x_i = x_j] \}$

ATM M having one w.t. + i.t. accepts D in time $O(n)$:

i.t. $\# x_1 \# x_2 \# \cdots \# x_m \#$



running time along path: $3n$

alternating pseudocode:

suppose $s_1, s_2, \dots, s_k$ are instructions; then, the following are instructions:

- doex begin $s_1; s_2; \dots; s_k$ end
- douniv begin $s_1; s_2; \dots; s_k$ end
- for $i \in M$ doex s
- for $i \in M$ douniv s

Example: chess

$Pos(P) =_{def}$ set of all positions reachable by active player in one move out of P

P is a win position for white

$$\Longleftrightarrow (\exists P_1 \in Pos(P))(\forall P_2 \in Pos(P_1))$$
$$(\exists P_3 \in Pos(P_2))(\forall P_4 \in Pos(P_3))$$
$$\dots$$
$$(\exists P_{2k+1} \in Pos(P_{2k}))[P_{2k+1} \text{ is checkmate for black}]$$

alternating program for win positions for white: $WP(P)$

if P is checkmate for white or draw then reject

else if white can move black into checkmate in one move then accept

else for $P' \in Pos(P)$ doex for $P'' \in Pos(P')$ douniv $WP(P'')$

multiT-ATIME (t) ; ... complexity classes like NTIME, ...

Proposition 1.

For each type τ and all $t: \mathbb{N} \to \mathbb{N}$, the following holds:

(1.) τ-NTIME $(t) \subseteq \tau$-ATIME (t)

(2.) τ-NSPACE $(t) \subseteq \tau$-ASPACE (t). $\blacksquare$

Theorem 2.

For each type τ, the following holds:

(1.) τ-ATIME (Pol t) = multiT-ATIME(Pol t) for $t(n) \geq n$

(2.) τ-ASPACE (s) = 2-T-ASPACE(s) for all $s(n) \geq \log n$. $\blacksquare$

Define:

ATIME (Pol t) $=_{def}$ multiT-ATIME (Pol t) for $t(n) \geq n$

ASPACE (s) $=_{def}$ 2-T-ASPACE (s) for $s(n) \geq \log n$

Proposition 3.

(1.) NTIME (Pol t) $\subseteq$ ATIME(Pol t) for $t(n) \geq n$

(2.) NSPACE $(s) \subseteq$ ASPACE (s) for $s(n) \geq \log n$ $\blacksquare$

diagram:

ASPACE (Pol t) $\overset{!}{=}$ DTIME $(2^{\text{Pol } t})$

|

ATIME (Pol t) $\overset{!}{=}$ DSPACE (Pol t) = NSPACE(Pol t)

|

NTIME (Pol t)

|

DTIME (Pol t)

e.g. ALIN = E
AL = P
AP = PSPACE

5.2 Alternating time

know: T-NTIME(s) $\subseteq$ DSPACE(s), s space-constructible, $s(n) \geq n$

Theorem 4.

For space-constructible functions $s(n) \geq n$,

$$\text{multiT-ATIME}(s) \subseteq \text{DSPACE}(s)$$

Proof: let M be a multiT-ATM accept set A in time s.
W.l.o.g. we may assume that M branches in two ways in each step, on stop M proceeds identically.

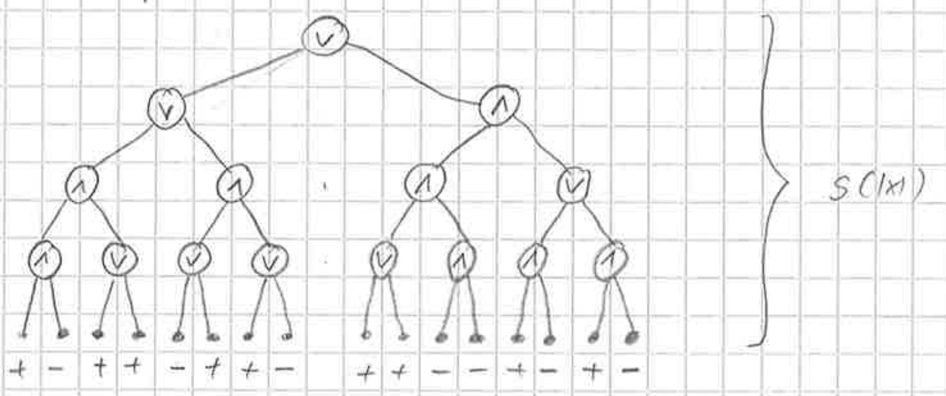
For input x, consider comp. paths of length $s(|x|)$, i.e., $\{0,1\}^{s(|x|)}$ is the set of all encodings of paths of M on x, $|z| = s(|x|)$. Define predicate

$$M(x,z) =_{\text{def}} \begin{cases} + & \text{if } M \text{ on } x \text{ along } z \text{ accepts } x \\ - & \text{otherwise} \end{cases}$$

For $|z| \leq s(|x|)$, define

$$\alpha(x,z) =_{\text{def}} \begin{cases} + & \text{if subtree of } \beta_M(x) \text{ rooted at } z \\ & \text{contains an accepting subtree} \\ - & \text{otherwise} \end{cases}$$

We have: M accepts x $\Leftrightarrow$ $\beta_M(x)$ contains acc. subtree $\Leftrightarrow \alpha(x,\varepsilon) = +$.



algorithm (based on preorder traversing):

$z := 0^{s(|x|)}$
$\alpha := M(x,z)$
while $z \neq \varepsilon$ **do** **begin**
    if $z = u0$ **then**
        if (u universal $\wedge\ \alpha = +$) **or** (u existential $\wedge\ \alpha = -$)
            then $z := u10^{s(|x|)-|u|-1}$
                $\alpha := M(x,z)$
            else $z := u$
    if $z = u1$ **then**
        $z := u$
end
if $\alpha = +$ **then** **accept** **else** **reject**

We only need to store z, α. Hence, running space is $s(|x|)$. Thus, $A \in DSPACE(s)$

Corollary 5.

For $s(n) \geq n$ s.t. $s(|x|) \in FDSPACE(Pol\ s)$,

$$ATIME(Pol\ s) \subseteq DSPACE(Pol\ s)$$

Savitch: $NSPACE(s) \subseteq DSPACE(s^2)$

Even: $NSPACE(s) \subseteq$ multiT-$ATIME(s^2)$!

Lemma 6.

Let $s(n) \geq n$; $t(n) \geq \log n$ such that $s(|x|), t(|x|) \in$ FmultiT-DTIME$(s \cdot t)$;
If a set A is accepted by a 2-T-NTM in space s and time 2^t then A is accepted by a multiT-ATM in time $O(s \cdot t)$.

Proof: Let M be a 2-T-NTM accepting A in space s and time 2^t. Let $K_1^x, K_2^x, \ldots, K_{2^{c \cdot s(|x|)}}^x$ be all config. of M on x.
W.l.o.g. we may assume that M proceeds identically on STOP, initial conf. is K_{ini}^x, uniquely. Define

$$R(x,i,j,\tau) =_{def} \begin{cases} 1 & \text{if } M \text{ on } x \text{ reaches } K_j \text{ from } \\ & K_i \text{ in } 2^\tau \text{ steps} \\ 0 & \text{otherwise} \end{cases}$$

We obtain: $x \in A \iff R(x, ini, acc, t(|x|)) = 1$

Idea: $R(x,i,j,\tau) = 1 \iff (\exists k)\left[R(x,i,k,\tau-1) = 1 \wedge R(x,k,j,\tau-1)\right]$

ATM M' computing $R(x, ini, acc, t(|x|))$:

(1.) $i := ini$; $j := acc$; $\tau = t(|x|)$
(2.) if $\tau = 0$ then if K_j^x is reachable from K_i^x in on step
(3.) then accept else reject
(4.) else for $k = 1$ to $2^{c \cdot s(|x|)}$ doex
(5.) douniv $R(x,i,k,\tau-1)$; $R(x,k,j,\tau-1)$

Computation tree $\beta_{M'}(x)$ has $2 \cdot t(|x|)$ levels; per level $O(s(|x|))$ steps.
Hence, each comp. path is $O(s(|x|) \cdot t(|x|))$.

Theorem 7.

(1.) $NSPACE(s) \subseteq multiT\text{-}ATIME(s^2)$ for $s(n) \geq n$ s.t.

2.) $s(|x|) \in F multiT\text{-}DTIME(s^2)$

(2.) $NSPACE(Pol\ s) \subseteq ATIME(Pol\ s)$ for $s(n) \geq n$ s.t.

$s(|x|) \in FDTIME(Pol\ s)$

Corollary 8.

$ATIME(Pol\ s) = DSPACE(Pol\ s)$ for $s(n) \geq n$ s.t.

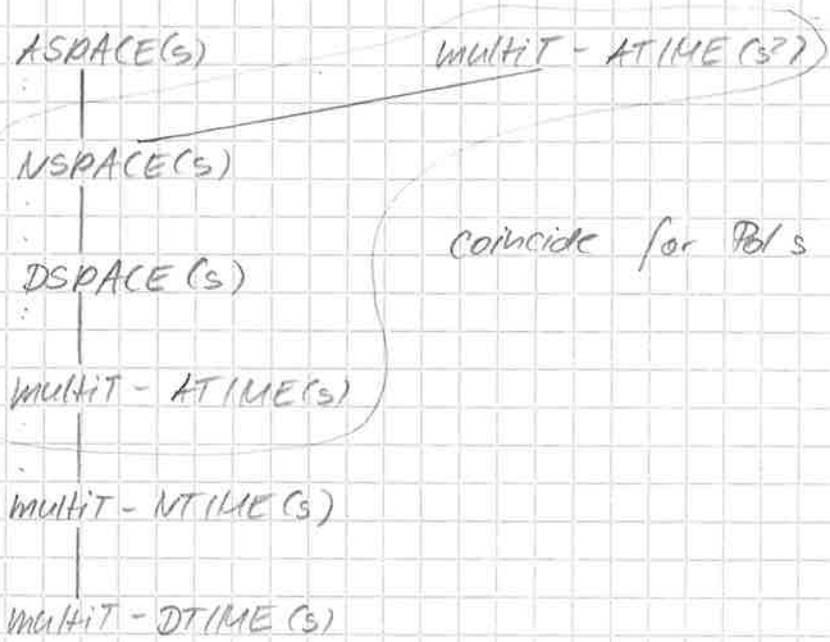
$s(|x|) \in FDTIME(Pol\ s)$

Corollary 9.

$NSPACE(s) \subseteq multiT\text{-}ATIME(s^2) \subseteq DSPACE(s^2)$

for $s(n) \geq n$ s.t. $s(|x|) \in F multiT\text{-}DTIME(s)$

Diagram:



5.3 Alternating space

Theorem 10

Let $s(n) \geq \log n$ be space-constructible. Then,

$$ASPACE(s) \subseteq DTIME(2^{O(s)}).$$

Proof: Let M be an ATM accepting L in space s.
Define M' to be that TM that simulates M on x as follows:

(1.) compute $s(|x|)$ (in time $2^{O(s)}$)

(2.) enumerate all config. of length $\leq s(|x|)$

(3.) label all accepting configurations

(4.) while there is a not-yet-labeled conf. k s.t.
k is existential and at least one successor configuration is labeled, or
k is universal and all successor configur. are labeled
do label first such configuration

(5.) if initial configuration is labeled then accept
else reject

Correctness: Only those conf. are labeled that evaluate to 1 in $\beta_M(x)$
We obtain:

$x \in L \iff M$ accepts x
$\iff \beta_M(x)$ contains acc. subtree
$\iff$ initial conf. evaluates to 1 in $\beta_M(x)$
$\iff M'$ accepts x

Running time: $2^{O(s)}$-iterations in (4) with $2^{O(s)}$ steps
Hence, $L \in DTIME(2^{O(s)})$.

Theorem 11.

Let $s(n) \geq \log n$ be space-constructible. Then,

$$DTIME(2^{O(s)}) \subseteq ASPACE(s).$$

Proof: Let M be a T-DTM accepting L in time $2^{c \cdot s(n)}$.

W.l.o.g. we may assume:

- M only uses on input $x = a_1 \ldots a_n$ cells $1, 2, \ldots, 2^{c \cdot s(n)}$
- M proceeds identically on STOP
- unique accepting conf.: tape cleaned, head on pos. 1, state z_1.

Consider the foll. triples (i, j, a), (i, j, sa) where $i \in \{0, \ldots, 2^{c \cdot s(n)}\}$, $j \in \{1, \ldots, 2^{c \cdot s(n)} + 1\}$, $a \in \Sigma$, s state. Define

(i, j, a) is correct $\Leftrightarrow_{df}$ M on x stores symbol a in cell j after step i, head elsewhere

(i, j, sa) is correct $\Leftrightarrow_{df}$ M on x stores symbol a in cell j after step i, head on j, state is s

Construct the set of correct triples inductively (s_0 initial state):

$$R_x =_{df} \{(0, 1, s_0 a_1), (0, 2, a_1), \ldots, (0, n, a_n), (0, n+1, \square), \ldots, (0, 2^{c \cdot s(n)}, \square)\}$$
$$\cup \{(i, 0, \square) \mid i \in \{0, 1, \ldots, 2^{c \cdot s(n)}\}\}$$
$$\cup \{(i, 2^{c \cdot s(n)} + 1, \square) \mid i \in \{0, 1, \ldots, 2^{c \cdot s(n)}\}\}$$

For $i \in \{1, \ldots, 2^{c \cdot s(n)}\}$, $j \in \{1, \ldots, 2^{c \cdot s(n)}\}$:

$(i-1, j-1, a)$, $(i-1, j, b)$, $(i-1, j+1, c)$ correct $\Rightarrow$ (i, j, b) correct (1)

$(i-1, j-1, sa)$, $(i-1, j, b)$, $(i-1, j+1, c)$ correct,

(i) $sa \rightarrow s'a' \begin{smallmatrix} L \\ O \end{smallmatrix}$ $\Rightarrow$ (i, j, b) correct (2)

(ii) $sa \rightarrow s'a'R$ $\Rightarrow$ $(i, j, s'b)$ correct (3)

$(i-1, j-1, a),\ (i-1, j, b),\ (i-1, j+1, sc)$ correct

(i) $sc \to s'c' \begin{smallmatrix} R \\ 0 \end{smallmatrix} \Rightarrow (i,j,b)$ correct (4)

(ii) $sc \to s'c'\ L \Rightarrow (i,j,s'b)$ correct (5)

$(i-1, j-1, a),\ (i-1, j, sb),\ (i-1, j+1, c)$ correct

(i) $sb \to s'b' \begin{smallmatrix} R \\ L \end{smallmatrix} \Rightarrow (i,j,b')$ correct (6)

(ii) $sb \to s'b'\ 0 \Rightarrow (i,j,s'b')$ correct (7)

For $u \in \{a, sa\}$, define

$$T(i,j,u) =_{df} \{ (t_1,t_2,t_3) \mid \text{correctness of } (i,j,u) \text{ follows from } (t_1,t_2,t_3) \text{ according to } (1),\dots,(7) \}$$

Clearly, $\| T(i,j,u) \| \leq$ const.

We obtain:

$x \in L \iff M$ accepts x

$\iff (2^{c \cdot s(n)}, 1, s, \square)$ is correct

ATM K for testing correctness of (i,j,u):

(1.) **if** $(i=0$ **or** $j=0$ **or** $j = 2^{c \cdot s(n)}+1)$

(2.) **then** **if** $(i,j,u) \in R_x$ **then** **accept** **else** **reject**

(3.) **else** **for** $(t_1,t_2,t_3) \in T(i,j,u)$ **do**$_\exists$

(4.) **for** $i \in \{1,2,3\}$ **do**$_{univ}$ $K(t_i)$

space: in each step only constant number of triples.

Thus, $|\text{triple}| \leq \underbrace{bin(i)}_{\leq c \cdot s(n)} + \underbrace{bin(j)}_{\leq c \cdot s(n)} +$ const.

Hence, $L \in ASPACE(s)$

Corollary 12.

$ASPACE(s) = DTIME(2^{O(s)})$ for space-constructible $s(n) \geq \log n$.

Diagram: