2.2 Syntax and semantics of first-order logic

In the following: Fix vocabulary τ

~~alphab~~ alphabet: $\Sigma(\tau) =_{def} \Sigma \cup \tau$

- vocabulary τ
- countable set of variables $\{x_0, x_1, \ldots, x_n, \ldots\}$
- connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- equation symbol $=$
- quantifier symbols $\exists, \forall$
- parentheses $($, $)$

Definition 5.

The set $T(\tau)$ of τ-terms is inductively defined as follows:

- base case:

  $c \in \tau$ ~~const~~ constant symbol $\Rightarrow$ $c \in T(\tau)$

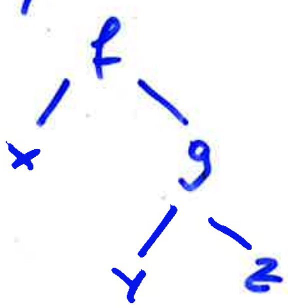
  $x_i \in \Sigma$ variable $\Rightarrow$ $x_i \in T(\tau)$

- inductive step:

  $t_1, \ldots, t_n \in T(\tau)$, $f \in \tau$ n-ary function symbol

  $\Rightarrow f t_1 \ldots t_n \in T(\tau)$

Example: $fxgyz$ represents $f(x, g(y, z))$



$+x1$ "=" $x+1$

Definition 6.

The set $FO(\tau)$ of τ-formulas is inductively defined as follows:

- base case (atomic formulas):

  $t_1, t_2 \in T(\tau) \Rightarrow (t_1 = t_2) \in FO(\tau)$

  $R \in \tau$ n-ary relation symbol, $t_1, \dots, t_n \in T(\tau)$

  $\Rightarrow R t_1 \dots t_n \in FO(\tau)$

- inductive step:

  $\varphi \in FO(\tau) \Rightarrow \neg\varphi \in FO(\tau)$

  $\varphi, \psi \in FO(\tau) \Rightarrow (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi),$
  $(\varphi \leftrightarrow \psi) \in FO(\tau)$

  $\varphi \in FO(\tau)$, $x_i \in \Sigma$ variable $\Rightarrow \exists x_i \varphi, \forall x_i \varphi \in FO(\tau)$

Example: let $\tau = \{E, f\}$ be a vocabulary; E binary relation symbol, f unary function symbol; formulas in $FO(\tau)$:

- $v_0 = v_1$
- $((E v_0 v_0 \vee f v_0 = v_1) \wedge \neg E v_1 f v_0)$
- $\forall v_0 \forall v_1 (\neg v_0 = v_1 \rightarrow E v_0 v_1)$
- $\forall v_0 \forall v_1 (E v_0 v_1 \rightarrow \exists v_2 (E v_0 v_2 \wedge E v_2 v_1))$

Variable x can be bound or free in $\varphi \in FO(\tau)$, e.g., in $\exists \underline{x} (E y z \wedge \forall \underline{z} (\underline{z} = \underline{x} \vee E y \underline{z}))$ underlined variables are bound; z is free and bound

Definition 7.

Let $t \in T(\tau)$ be a term, let $\varphi \in FO(\tau)$ be a formula. Let $var(t)$, $var(\varphi)$ denote the set of variables of t, φ.

The set $free(\varphi)$ of ~~free va~~ free variables of φ is inductively defined as follows:

- base case: if φ is atomic then $free(\varphi) =_{def} var(\varphi)$
- inductive step:

$$free(\neg\varphi) =_{def} free(\varphi)$$

$$free(\varphi \circ \psi) =_{def} free(\varphi) \cup free(\psi) \text{ for } \circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$$

$$free(\exists x \varphi) = free(\forall x \varphi) =_{def} free(\varphi) \setminus \{x\}$$

A formula $\varphi \in FO(\tau)$ is called a theorem iff $free(\varphi) = \emptyset$.

$FO_0(\tau) =_{def} \{\varphi \in FO(\tau) \mid free(\varphi) = \emptyset\}$

Definition 8.

Let τ be a vocabulary.

A pair $\mathcal{I} = (\mathfrak{A}, \beta)$ is said to be a τ-interpretation if and only if

(1) $\mathfrak{A}$ is a τ-structure $(\mathfrak{A} = (A, \alpha))$

(2) x_i variable $\longrightarrow \beta(x_i) \in A$ (assignment)

Definition 9.

Let τ be a vocabulary. Let $\mathcal{I} = (\mathfrak{A}, \beta)$ be a τ-interpretation, $\mathfrak{A} = (A, \alpha)$ is a τ-structure.

(1.) The interpretation $[\![t]\!]^{\mathcal{I}}$ of a term $t \in T(\tau)$ is inductively defined as follows:

- $[\![c]\!]^{\mathcal{I}} =_{def} \alpha(c)$ for all const. symbols $c \in T(\tau)$
- $[\![x_i]\!]^{\mathcal{I}} =_{def} \beta(x_i)$ for all variables $x_i \in T(\tau)$
- $[\![f\, t_1 \ldots t_n]\!]^{\mathcal{I}} =_{def} \alpha(f)([\![t_1]\!]^{\mathcal{I}}, \ldots, [\![t_n]\!]^{\mathcal{I}})$ for all n-ary function symb. $f \in \tau$

(2.) The interpretation $[\![\varphi]\!]^{\mathcal{I}}$ of a formula $\varphi \in FO(\tau)$ is inductively defined as follows:

- base case:

$$[\![t_1 = t_2]\!]^{\mathcal{I}} =_{def} \begin{cases} 1 & \text{if } [\![t_1]\!]^{\mathcal{I}} = [\![t_2]\!]^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$$

$$[\![Rt_1 \dots t_n]\!]^{\mathcal{I}} =_{def} \begin{cases} 1 & \text{if } ([\![t_1]\!]^{\mathcal{I}}, \dots, [\![t_n]\!]^{\mathcal{I}}) \in \alpha(R) \\ 0 & \text{otherwise} \end{cases}$$

- inductive step:

$$[\![\neg\varphi]\!]^{\mathcal{I}} =_{def} \text{non}\,([\![\varphi]\!]^{\mathcal{I}})$$
$$[\![\varphi \wedge \psi]\!]^{\mathcal{I}} =_{def} \text{et}\,([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$$
$$[\![\varphi \vee \psi]\!]^{\mathcal{I}} =_{def} \text{vel}\,([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$$
$$[\![\varphi \rightarrow \psi]\!]^{\mathcal{I}} =_{def} \text{seq}\,([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$$
$$([\![\varphi \leftrightarrow \psi]\!]^{\mathcal{I}} =_{def} \text{aeq}\,([\![\varphi]\!]^{\mathcal{I}}, [\![\psi]\!]^{\mathcal{I}})$$
$$[\![\exists x_i \varphi]\!]^{\mathcal{I}} =_{def} \max_{\mathcal{I}' \overset{x_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$
$$[\![\forall x_i \varphi]\!]^{\mathcal{I}} =_{def} \min_{\mathcal{I}' \overset{x_i}{=} \mathcal{I}} [\![\varphi]\!]^{\mathcal{I}'}$$

Here, $(\mathfrak{A}_1, \beta_1) \overset{x}{=} (\mathfrak{A}_2, \beta_2)$ means $\mathfrak{A}_1 = \mathfrak{A}_2$ and $\beta_1(y) = \beta_2(y)$ for $x \neq y$.

Example: $\tau_{Ar}^{<} = \{0, 1, +, <\}$, $\varphi = \forall x (\overset{<0x}{0<x} \rightarrow \exists y (\overset{+y1=x}{y+1=x}))$

(1.) $\mathcal{I} = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (\mathbb{N}, \alpha)$:

$\alpha(0) =_{def} 0$ $(\in \mathbb{N})$

$\alpha(1) =_{def} 1$ $(\in \mathbb{N})$

$\alpha(+) =_{def} +$ (addition on $\mathbb{N}$)

$\alpha(<) =_{def} <$ (less-than on $\mathbb{N}$)

$\beta(x) =_{def} 9$ (irrelevant)

$\beta(y) =_{def} 20$ (irrelevant)

We obtain:

$$[\![\varphi]\!]^{\mathcal{I}} = \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} [\![0 < x \to \exists y\,(y+1=x)]\!]^{\mathcal{I}'}$$

$$= \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} seq\left([\![0<x]\!]^{\mathcal{I}'}, [\![\exists y\,(y+1=x)]\!]^{\mathcal{I}'} \right)$$

$$= \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} seq\left([\![0<x]\!]^{\mathcal{I}'}, \max_{\mathcal{I}'' \overset{y}{=} \mathcal{I}'} [\![y+1=x]\!]^{\mathcal{I}''} \right)$$

$$= \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} seq\left([\![0<x]\!]^{\mathcal{I}'}, \max_{\mathcal{I}'' \overset{y}{=} \mathcal{I}'} [\![y+1]\!]^{\mathcal{I}''} = [\![x]\!]^{\mathcal{I}''} \right)$$

$$= \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} seq\left([\![0<x]\!]^{\mathcal{I}'}, \max_{\mathcal{I}'' \overset{y}{=} \mathcal{I}'} [\![y]\!]^{\mathcal{I}''} + \alpha(1) = [\![x]\!]^{\mathcal{I}''} \right)$$

$$= \min_{\mathcal{I}' \overset{x}{=} \mathcal{I}} seq\left([\![0<x]\!]^{\mathcal{I}'}, \max_{n \in \mathbb{N}} n+1 = [\![x]\!]^{\mathcal{I}'} \right)$$

$$= \min_{m \in \mathbb{N}} seq\left(0<m, \max_{n \in \mathbb{N}} n+1 = m \right)$$

$$= \min_{m \in \mathbb{N}} seq\left(0<m, \begin{cases} 1 & \text{if } m>0 \\ 0 & \text{if } m=0 \end{cases} \right)$$

$$= \min \{ \underbrace{seq(0,0)}_{m=0}, \underbrace{seq(1,1)}_{m>0} \}$$

$$= \min \{1, 1\}$$

$$= 1$$

(2.) $I = (\mathcal{A}, \beta)$, $\mathcal{A} = (\mathbb{R}_{\geq 0}, \alpha)$:

$\alpha(0) =_{def} 0 \quad (\in \mathbb{R}_{\geq 0})$

$\alpha(1) =_{def} 1 \quad (\in \mathbb{R}_{\geq 0})$

$\alpha(+) =_{def} +$ (addition on $\mathbb{R}_{\geq 0}$)

$\alpha(<) =_{def} <$ (less-than on $\mathbb{R}_{\geq 0}$)

$\beta(x) =_{def} \pi$ (irrelevant)

$\beta(y) =_{def} e$ (irrelevant)

We obtain:

$$[\![\varphi]\!]^I = \min_{m \in \mathbb{R}_{\geq 0}} \operatorname{seq}\left(0 < m, \max_{n \in \mathbb{R}_{\geq 0}} n + 1 = m\right)$$

$$= \min_{m \in \mathbb{R}_{\geq 0}} \operatorname{seq}\left(0 < m, \begin{cases} 1 & \text{if } m \geq 1 \\ 0 & \text{if } 0 \leq m < 1 \end{cases}\right)$$

$$= \min \{ \underbrace{\operatorname{seq}(0,0)}_{m=0}, \underbrace{\operatorname{seq}(1,0)}_{0<m<1}, \underbrace{\operatorname{seq}(1,1)}_{m \geq 1} \}$$

$$= \min \{1, 0, 1\}$$

$$= 0$$

(3.) $I = (\mathcal{A}, \beta)$, $\mathcal{A} = (\mathbb{N}, \alpha)$:

$\alpha(0) =_{def} 2 \quad (\in \mathbb{N})$

$\alpha(1) =_{def} 2 \quad (\in \mathbb{N})$

$\alpha(+) =_{def} \cdot$ (multiplication on $\mathbb{N}$)

$\alpha(<) =_{def} \mid$ (divisibility on $\mathbb{N}$)

$\beta(x) =_{def} 11$

$\beta(y) =_{def} 28$

$\min_{n \in \mathbb{N}} seq(2|m, \max_{n \in \mathbb{N}} n \cdot 2 = m)$

$\min_{n \in \mathbb{N}} seq(2|m, \begin{cases} 1 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases})$

$\min \{ \underset{m \text{ even}}{seq(1,1)}, \underset{m \text{ odd}}{seq(0,0)} \}$

$\min \{1, 1\}$

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and proofs in first-order logic

$\in FO(\tau)$

... $\subseteq FO(\tau)$

interpretations

vocabulary, let I be a τ-interpretation.
said to be a model of $\varphi \in FO(\tau)$
only if $[\![\varphi]\!]^I = 1$.