2.4 Limits of first-order logic

2.4.1 Excursion: Compatability theory

(TM = JAVA = PART = RAM)

We use RAM (random access machines) as a comp. modul:

- countably many registers R0, R1, R2, ..., each register R_i containing a number $\langle R_i \rangle \in \mathbb{N}$
- an instruction register BR containing the next instruction $\langle BR \rangle$ to be executed
- a finite instruction set with instructions of several types:

type	syntax	semantics
transport	$R_i \leftarrow R_j$	$\langle R_i \rangle := \langle R_j \rangle$, $\langle BR \rangle := \langle BR \rangle + 1$
	$RR_i \leftarrow R_j$	$\langle R \langle R_i \rangle \rangle := \langle R_j \rangle$, $\langle BR \rangle := \langle BR \rangle + 1$
	$R_i \leftarrow RR_j$	$\langle R_i \rangle := \langle R \langle R_j \rangle \rangle$, $\langle BR \rangle := \langle BR \rangle + 1$
arithmetic	$R_i \leftarrow k$	$\langle R_i \rangle := k$, $\langle BR \rangle := \langle BR \rangle + 1$
	$R_i \leftarrow R_j + R_k$	$\langle R_i \rangle := \langle R_j \rangle + \langle R_k \rangle$, $\langle BR \rangle := \langle BR \rangle + 1$
	$R_i \leftarrow R_j - R_k$	$\langle R_i \rangle := \max\{\langle R_j \rangle - \langle R_k \rangle, 0\}$, $\langle BR \rangle := \langle BR \rangle + 1$
jumps	GOTO k	$\langle BR \rangle := k$
	IF $R_i = 0$ GOTO k	$\langle BR \rangle := \begin{cases} k & \text{if } \langle R_i \rangle = 0 \\ \langle BR \rangle + 1 & \text{otherwise} \end{cases}$
stop	STOP	$\langle BR \rangle := 0$

- a program consisting of $m \in \mathbb{N}$ instructions enumerated by [1], [2], ..., [m]

The input $(x_1, \dots, x_m) \in \mathbb{N}^m$ is given by the following initial configuration:

$$\langle Ri \rangle := x_{i+1} \quad \text{for } 0 \le i \le m-1$$
$$\langle Ri \rangle := 0 \quad \text{for } i \ge m$$

A RAM computation stops when BR contains 0. Then, the output is given by $\langle R0 \rangle$.

Example: RAM M for computing $\text{mult}: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$: $(x, y) \mapsto x \cdot y$:

```
[1]  R3 ← 1
[2]  IF R1 = 0  GOTO 6
[3]  R2 ← R2 + R0
[4]  R1 ← R1 - R3
[5]  GOTO 2
[6]  R0 ← R2
[7]  STOP
```

Definition 38.

A function $f: \mathbb{N}^n \to \mathbb{N}$ is said to be computable (or partial recursive) iff there ex. a RAM that, on input $(x_1, \dots, x_n)$,

(i) stops and contains $f(x_1, \dots, x_n)$ in R0, if $f(x_1, \dots, x_n)$ is defined,

(ii) does not stop, otherwise.

Remark: Class of computable functions independent of chosen computational model (Church-Turing thesis); in particular, we may restrict RAMs to +1, −1 and not allowing indirect addressing

Definition 39.

(1.) A partial recursive function f is called (total) recursive iff f is total

(2.) A set $A \subseteq \mathbb{N}^n$ is called decidable iff characteristic function $c_A : \mathbb{N}^n \to \{0,1\}$ is total recursive.

(3.) A set $A \subseteq \mathbb{N}^n$ is called enumerable iff $A = D_f$ (domain of f) for some partial recursive function f.

Theorem 40.

Let $A \subseteq \mathbb{N}$, $A \neq \emptyset$. The following statements are equivalent:

(1.) A is enumerable

(2.) $A = R_f$ (range of f) for some part. rec. function f

(3.) $A = R_f$ for some total rec. function f

(4.) A is accepted by an NRAM.

Proof:

(1) $\Rightarrow$ (4): Let A be enumerable, i.e., $A = D_f$ for some partial rec. f computed by RAM M.

Define N to be that RAM that on input $n \in \mathbb{N}$

(i) guesses a number $t \in \mathbb{N}$ of steps

(ii) runs M on n for at most t steps

(iii) accepts n iff M terminates within t steps

Clearly, $n \in A$ iff N accepts n on some comp. path

(4) $\Rightarrow$ (3): Let A be accepted by NRAM M. Fix $n_0 \in A$.

Define $f: \mathbb{N}^2 \to \mathbb{N}$ to be the foll. function

$$f(n,\beta) =_{def} \begin{cases} n & \text{if } \beta \text{ is an encoding of an acc. comp. path of } M \text{ on } n \\ n_0 & \text{otherwise} \end{cases}$$

Clearly, f is total recursive and $A = R_f$.

(3) $\Rightarrow$ (2): Trivial

(2) $\Rightarrow$ (1): Let $A = R_f$ for some partial rec. funct. $f: \mathbb{N} \to \mathbb{N}$.

Define $g: \mathbb{N} \to \mathbb{N}$ to be the foll. funct.:

$$g(n) =_{def} \begin{cases} x & \text{if } x \text{ is minimal subject to } f(x) = n \\ \text{n.d.} & \text{if there is no } x \text{ s.t. } f(x) = n \end{cases}$$

Then, g is partial recursive (dove-tailing) and $D_g = R_f$. ∎

Theorem 4.1.

(1.) A decidable $\Leftrightarrow$ $A, \overline{A}$ enumerable

(2.) A enumerable $\Leftrightarrow$ $A = \text{proj}(B)$ for some decidable B. ∎

Note: $\text{proj}(B) =_{def} \{ x \mid (\exists z)[(x,z) \in B] \}$

Theorem 4.2.

There exists an enumerable set that is not decidable. ∎

Definition 4.3. (Arithmetical hierarchy, Kleene-Mostowski)

Let $k \in \mathbb{N}$, $A \subseteq \mathbb{N}$.

(1.) $A \in \Sigma_k \Leftrightarrow_{def}$ there ex. a decidable set B s.t.

$$x \in A \Leftrightarrow \exists z_1 \forall z_2 \ldots Q z_k \, ((x, z_1, \ldots, z_k) \in B)$$

(2.) $A \in \Pi_k \Leftrightarrow_{def}$ there ex. a decidable set B s.t.

$$x \in A \Leftrightarrow \forall z_1 \exists z_2 \ldots Q z_k \, ((x, z_1, \ldots, z_k) \in B)$$

(3.) $AH =_{def} \bigcup_{k \geq 0} (\Sigma_k \cup \Pi_k)$

Lemma 44.

(1.) $\Sigma_0 = \Pi_0 = REC$ (~~set of~~ class of decidable sets)

(2.) $\Sigma_1 = RE$ (class of enumerable sets)

(3.) $A \in \Sigma_k \Leftrightarrow \bar{A} \in \Pi_k$ for $k \in \mathbb{N}$

(4.) $A \in \Sigma_{k+1} \Leftrightarrow A = \mathrm{proj}(B)$ for some $B \in \Pi_k$

(5.) $\Sigma_k \cup \Pi_k \subseteq \Sigma_{k+1} \cap \Pi_{k+1}$

(6.) $\Sigma_k \setminus \Pi_k \neq \emptyset$ and $\Pi_k \setminus \Sigma_k \neq \emptyset$ for $k \geq 1$.

Diagrams:

$$\begin{tikzcd}
& AH & \\
& \vdots & \\
\Sigma_3 & & \Pi_3 \\
\Sigma_2 \arrow[u, dash] \arrow[urr, dash] & & \Pi_2 \arrow[u, dash] \arrow[ull, dash] \\
RE = \Sigma_1 \arrow[u, dash] \arrow[urr, dash] & & \Pi_1 \arrow[u, dash] \arrow[ull, dash] \\
& \Sigma_0 = \Pi_0 = REC \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

2.4.2 Undecidability of first-order logic

Let τ be a countable vocabulary.
Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be all $FO(\tau)$-formulas in quasi-lexicographic order; encode formulas by numbers: $\langle \varphi_i \rangle =_{def} i$, i.e., $\varphi_{\langle\varphi\rangle} = \varphi$

Consider

$$FO_t(\tau) =_{def} \{ \varphi \mid \varphi \in FO(\tau), \varphi \text{ is true} \}$$

Theorem 45. (Church 1936)

Let τ be a vocabulary containing a constant symbol, a unary function symbol, and for all $n \in \mathbb{N}$, an n-ary relation symbol. Then, $FO_t(\tau)$ is undecidable.

Remark: Thm. 45 holds true for many finite vocabularies (e.g., only n-ary relation symbols for $n=2,3$).

$FO_t(\tau)$ is decidable	lower bound	upper b.
$\tau = \emptyset$	$NSPACE(s), s = o(\sqrt{n})$	$DSPACE(n)$
τ: only unary rel.	$DSPACE\left(2^{2^{c \cdot \frac{n}{\log n}}}\right)$	?
τ: only unary funct.	$DSPACE\left(2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}c\cdot\log n\right)$	$DSPACE\left(2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}d\cdot n\right)$

Proof: We assign a formula $\varphi_i \in FO(\tau)$ to $i \in \mathbb{N}$ effectively (i.e., via an alg.) s.t.

$$i \in K_0 \iff \varphi_i \text{ true} \iff \varphi_i \in FO_t(\tau)$$

Hence, $FO_t(\tau)$ is undecidable.

Suppose we are given $i \in \mathbb{N}$. RAM M_i consists of instructions $\beta_0, \beta_1, \dots, \beta_k$; w.l.o.g. β_k is the only STOP instruction and each comp. uses β_k to stop. Choose $m \in \mathbb{N}$ so that for each occurring register R_i, $i \leq m$. (No indirect addressing!)
If $\langle BR \rangle = b$, $\langle R_i \rangle = a_i$ for $i \in \{0, 1, \dots, m\}$ after step t then the tuple $(b, a_0, a_1, \dots, a_m)$ is the configuration of M_i after t steps. For $t \in \mathbb{N}$, define

$$C_t =_{def} \text{configuration of } M_i \text{ on input } i \text{ after } t \text{ steps}$$

In particular, $C_0 = (0, i, 0, \dots, 0)$.
We have:

$$i \in K_0 \iff \text{there ex. } t, a_0, a_1, \dots, a_m \text{ such that } C_t = (\beta_k, a_0, a_1, \dots, a_m)$$

Choose the following symbols from τ:
- $(m+3)$-arg relation symbol R
- binary relation symbol $<$
- unary function symbol s
- constant symbol c

Define the following terms:

$$\overline{0} =_{def} c$$

$$\overline{j+1} =_{def} s(\overline{j})$$

$$\overline{C_t} =_{def} (\overline{b}, \overline{a_0}, \dots, \overline{a_m}) \quad \text{for } C_t = (b, a_0, a_1, \dots, a_m)$$

Define τ-structure $\mathfrak{A}_i =_{def} (\mathbb{N}, \alpha)$ where

$$\alpha(R) =_{def} \{ (t, C_t) \mid t \in \mathbb{N} \}$$

$$\left.\begin{array}{l} \alpha(s)(x) =_{def} x+1 \\ \alpha(c) =_{def} 0 \end{array}\right\} \quad \alpha(\overline{j}) = j$$

$$\alpha(<) =_{def} <_{\mathbb{N}}$$

We construct a formula ψ_i such that

(1.) $\mathfrak{A}_i$ is a model of ψ_i

(2.) if $\mathfrak{A}$ is a model of ψ_i then $\mathfrak{A}$ is a model of $R(\overline{t}, \overline{C_t})$ for all $t \in \mathbb{N}$

Using ψ_i, define

$$\varphi_i =_{def} \psi_i \rightarrow \exists t \exists x_0 \dots \exists x_m \, R(t, \overline{\beta_k}, x_0, \dots, x_m)$$

Claim: φ_i true $\Longleftrightarrow$ $i \in K_0$

$\boxed{\Rightarrow}$ Suppose φ_i is true. Since $\mathfrak{A}_i$ is a model of ψ_i (by (1.)), $\mathfrak{A}_i$ is a model of $\exists t \exists x_0 \dots \exists x_m \, R(t, \overline{\beta_k}, x_0, \dots, x_m)$

Hence, there ex. $t, a_0, \dots, a_m \in \mathbb{N}$ s.t. $\alpha(R)(t, \beta_k, a_0, \dots, a_m) = 1$

Hence, there ex. $t, a_0, \dots, a_m \in \mathbb{N}$ s.t. $C_t = (\beta_k, a_0, \dots, a_m)$

Hence, $i \in K_0$.

$\boxed{\Leftarrow}$ Suppose $i \in K_0$. Then, there ex. $t, a_0, \dots, a_m \in \mathbb{N}$ s.t. $C_t = (\beta_k, a_0, \dots, a_m)$. Assume $[\![\varphi_i]\!]^{\mathfrak{a}} = 1$.

Then, by (2.), $[\![R(\bar{t}, \overline{\beta_k}, \bar{a}_0, \dots, \bar{a}_m)]\!]^{\mathfrak{a}} = 1$.

Hence, $[\![\exists t \exists x_0 \dots \exists x_m \; R(t, \overline{\beta_k}, x_0, \dots, x_m)]\!]^{\mathfrak{a}} = 1$

Hence, φ_i is true

It remains to construct φ_i:

$$\eta =_{def} \forall x \forall y \, (x < y \vee x = y \vee y < x)$$ $<$ is linear order

$$\wedge \forall x \, (c = x \vee c < x)$$ c is minimal w.r.t. $<$

$$\wedge \forall x \, (x < s(x) \wedge \forall y \, (x < y \to (y = s(x) \vee s(x) < y)))$$

$s(x)$ is successor of x w.r.t. $<$

That is, if $\mathfrak{A} = (A, \alpha)$ is a model of η then $(\mathbb{N}, 0, <_{\mathbb{N}})$ is isomorphic to some initial segment of $(A, \alpha(0), \alpha(<))$.

For each instruction β_j we define a formula γ_j:

(1.) $\beta_j = Rk \leftarrow Rl$:

$$\gamma_j =_{def} \forall t \forall x_0 \dots \forall x_m \, (R(t, \bar{j}, x_0, \dots, x_m) \to R(s(t), \overline{j+1}, x_0, \dots, x_{k-1}, x_l, x_{k+1}, \dots, x_m))$$

(2.) $\beta_j = Rk \leftarrow (Rl)+1$

$$\gamma_j =_{def} \forall t \forall x_0 \dots \forall x_m \, (R(t, \bar{j}, x_0, \dots, x_m) \to R(s(t), \overline{j+1}, x_0, \dots, x_{k-1}, s(x_l), x_{k+1}, \dots, x_m))$$

(3.) $\beta_j = Rk \leftarrow Rl - 1$

$\gamma_j :=_{def} \forall t \, \forall x_0 \ldots \forall x_m \, (R(t, \bar{j}, x_0, \ldots, x_m)$

$\rightarrow ((x_\ell = \bar{0} \wedge R(s(t), \overline{j+1}, x_0, \ldots, x_{\ell-1}, x_\ell, x_{\ell+1}, \ldots, x_m))$

$\vee (\neg x_\ell = \bar{0} \wedge$

$\exists u \, (s(u) = x_\ell \wedge$

$R(s(t), \overline{j+1}, x_0, \ldots, x_{\ell-1}, u, x_{\ell+1}, \ldots, x_m))))$

(4.), (5.) Exercise

Finally, $\psi_i :=_{def} \eta \wedge R(\bar{0}, \bar{1}, \bar{0}, \ldots, \bar{0}) \wedge \gamma_0 \wedge \gamma_1 \wedge \ldots \wedge \gamma_{k-1}$.

2.4.3 Gödel's incompleteness theorem

$\tau_{Ar} = \{+, \cdot, 0, 1\}$, $\mathcal{N}$ τ_{Ar}-structure $(\mathbb{N}, +, \cdot, 0, 1)$

Define

$$Th(\mathcal{N}) =_{def} \{ \varphi \in FO_0(\tau_{Ar}) \mid [\![\varphi]\!]^{\mathcal{N}} = 1 \}$$

$Th(\mathcal{N})$ is called elementary arithmetic.

Gödel 1931: There is no finite $\Phi \subset FO(\tau_{Ar})$ s.t. $\Phi^{\models} = Th(\mathcal{N})$

Define $\bar{0} =_{def} 0$, $\overline{k+1} = \bar{k} + 1$

Definition 49.

(1.) A relation $R \subseteq \mathbb{N}^n$ is said to be expressible in Φ if and only if there ex. $\varphi \in FO(\tau_{Ar})$ s.t.

(i) $(a_1, \dots, a_n) \in R \Rightarrow \varphi(\bar{a}_1, \dots, \bar{a}_n) \in \Phi^{\models}$

(ii) $(a_1, \dots, a_n) \notin R \Rightarrow \neg\varphi(\bar{a}_1, \dots, \bar{a}_n) \in \Phi^{\models}$

(2.) A function $f: \mathbb{N}^n \to \mathbb{N}$ is said to be expressible in Φ if and only if there ex. $\varphi \in FO(\tau_{Ar})$ s.t.

(i) $f(a_1, \dots, a_n) = b \Rightarrow \varphi(\bar{a}_1, \dots, \bar{a}_n, \bar{b}) \in \Phi^{\models}$

(ii) $f(a_1, \dots, a_n) \neq b \Rightarrow \neg\varphi(\bar{a}_1, \dots, \bar{a}_n, \bar{b}) \in \Phi^{\models}$

(iii) $((\varphi(\bar{a}_1, \dots, \bar{a}_n, \bar{b}) \land \varphi(\bar{a}_1, \dots, \bar{a}_n, \bar{c})) \to \bar{b} = \bar{c}) \in \Phi^{\models}$

Example: The following are expressible in $Th(\mathfrak{N})$:

1. $x<y$: $\varphi(x,y) =_{def} \neg x=y \wedge \exists z(x+z=y)$
2. $x|y$: $\varphi(x,y) =_{def} \exists z(\neg z=0 \wedge x\cdot z=y)$
3. $s: x \mapsto x+1$: $\varphi(x,y) =_{def} y=x+1$

Note that: c_R expressible in $\Phi \implies R$ expressible in Φ

Proposition 50.

Let Φ be a complete theory. A function f is expressible in Φ via $\varphi \in FO(\tau_{Ar})$ if and only if

$$f(a_1,\dots,a_n)=b \iff \varphi(\bar{a}_1,\dots,\bar{a}_n,\bar{b}) \in \Phi$$

■

Proposition 51.

If Φ is consistent and decidable then each relation expressible in Φ is decidable and each function expressible in Φ is computable.

Proof: (only for unary functions)
Let $f: \mathbb{N} \to \mathbb{N}$ be expressible in Φ via φ. That is,
if $f(n)$ is defined then $\varphi(\bar{n}, \overline{f(n)}) \in \Phi^{\vdash}$;
if $m \neq f(n)$ then $\neg\varphi(\bar{n},\bar{m}) \in \Phi^{\models}$ (and thus,
$\varphi(\bar{n},\bar{m}) \notin \Phi^{\models}$ by consistency)

Since Φ is decidable, $\Phi^{\models} = \Phi^{\vdash}$ is enumerable.
By thm. 40, $\Phi^{\models} = R_g$ for some total recursive g.

Algorithm for comp. $f(n)$: Compute $g(0), g(1), g(2), \ldots$; if $g(x) = \varphi(\bar{n}, \bar{m})$ output m. ■

Lemma 52.

There is a function $\beta : \mathbb{N}^3 \to \mathbb{N}$ expressible in $Th(\mathcal{N})$ s.t. for all finite sequences $a_0, a_1, \ldots, a_r \in \mathbb{N}$, there ex. $s, p \in \mathbb{N}$ satisfying

$$\beta(s, p, i) = a_i$$

for all $i \in \{0, 1, \ldots, r\}$.

Proof: Let $a_0, a_1, \ldots, a_r \in \mathbb{N}$. Choose prime number $p > a_0, a_1, \ldots, a_r, r+1$. Define

$$s =_{def} \sum_{j=0}^{r} (j+1 + a_j p)\, p^{2j}$$

That is, $1\ a_0\ 2\ a_1\ 3\ a_2 \ldots a_{r-1}\ r{+}1\ a_r$ is p-adic encoding of s

$(i+1, a_i)$

Claim: $a = a_i \iff$ there ex. $b_0, b_1, b_2 \in \mathbb{N}$ s.t.

(1) $s = b_0 + b_1(i+1 + ap + b_2 p^2)$

(2) $a < p$

(3) $b_0 < b_1$

(4) $b_1 = p^{2m}$ for some m

$\boxed{\Rightarrow}$ $b_0 = \sum_{j=0}^{i-1} (j+1 + a_j p)\, p^{2j}$

$b_1 = p^{2i}$

$b_2 = \sum_{j=i+1}^{r} (j+1 + a_j p)\, p^{2(j-i-1)}$

$\boxed{\Leftarrow}$ Δ

Now, (4) is equivalent to (4'):

$$\exists c\,(c\cdot c = b_1 \wedge \forall d((\neg d = 1 \wedge d \mid b_1) \to p \mid d))$$

Thus, $\beta(s,p,i) = a \iff \exists b_0 \exists b_1 \exists b_2\,((1) \wedge (2) \wedge (3) \wedge (4'))$. ∎

<u>Example:</u> $exp(x,y) =_{def} x^y$ expressible in $Th(\mathcal{N})$.
Use formula $\varphi \in FO(\tau_{Ar})$ expressing β as in Lemma 52, i.e.,

$$\beta(s,p,i) = a \iff \varphi(\bar{s},\bar{p},\bar{i},\bar{a}) \in Th(\mathcal{N})$$

We have:

$$exp(x,y) = z$$

$$\iff \exists s \exists p\,(\beta(s,p,0) = 1 \wedge \beta(s,p,y) = z \wedge$$
$$\forall i\,(0 \le i < y \to \beta(s,p,i)\cdot x = \beta(s,p,i+1))$$

$$\iff \exists s \exists p\,(\varphi(s,p,\bar{0},\bar{1}) \wedge \varphi(s,p,\bar{y},\bar{z}) \wedge$$
$$\forall i\,(0 \le i < y \to \exists u\,(\varphi(s,p,i,u) \wedge \varphi(s,p,i+\bar{1},u\cdot\bar{x}))))$$

<u>Theorem 53.</u>

(1.) Each decidable set is expressible in $Th(\mathcal{N})$

(2.) Each computable function is expressible in $Th(\mathcal{N})$ ∎

<u>Theorem 54.</u> (Fixed-point theorem)

Suppose all comp. functions are expressible in $\Phi \subseteq FO(\tau_{Ar})$. Then, for each τ_{Ar}-formula $\psi(x)$ there ex. a sentence $\varphi \in FO_0(\tau_{Ar})$ s.t.

$$\Phi \models \varphi \leftrightarrow \psi(\langle\varphi\rangle)$$

Proof: Define $f(n) =_{def} \varphi_n(x_1|\bar{n}, \dots, x_m|\bar{n})$
$(=_{def} \varphi_n(\bar{n}, \dots, \bar{n}))$

Then, f is computable. So, f is expressed in Φ via η.

Define $\gamma(x) =_{def} \forall z(\eta(x,z) \to \psi(z))$

$$\varphi =_{def} \forall z(\eta(\overline{\langle \gamma \rangle}, z) \to \psi(z))$$

That is, $\varphi = \gamma(\overline{\langle \gamma \rangle})$.

We obtain: $f(\langle \gamma \rangle) = \varphi_{\langle \gamma \rangle}(\overline{\langle \gamma \rangle}, \dots, \overline{\langle \gamma \rangle}) = \gamma(\overline{\langle \gamma \rangle}) = \varphi$

Hence, $\Phi \vDash \eta(\overline{\langle \gamma \rangle}, \overline{\langle \varphi \rangle})$ (*)

We show: $\Phi \vDash \varphi \leftrightarrow \psi(\overline{\langle \varphi \rangle})$

$\boxed{\to}$ $\Phi \cup \{\varphi\} \vDash \varphi = \forall z(\eta(\overline{\langle \gamma \rangle}, z) \to \psi(z))$.

Thus, $\Phi \cup \{\varphi\} \vDash \eta(\overline{\langle \gamma \rangle}, \overline{\langle \varphi \rangle}) \to \psi(\overline{\langle \varphi \rangle})$

By (*), $\Phi \cup \{\varphi\} \vDash \psi(\overline{\langle \varphi \rangle})$

Hence, $\Phi \vDash \varphi \to \psi(\overline{\langle \varphi \rangle})$

$\boxed{\leftarrow}$ $\Phi \vDash \forall z \forall z'((\eta(\overline{\langle \gamma \rangle}, z) \wedge \eta(\overline{\langle \gamma \rangle}, z') \to z = z')$

By (*), $\Phi \vDash \forall z(\eta(\overline{\langle \gamma \rangle}, z) \to z = \overline{\langle \varphi \rangle})$

Hence, $\Phi \cup \{\psi(\overline{\langle \varphi \rangle})\} \vDash \underbrace{\forall z(\eta(\overline{\langle \gamma \rangle}, z) \to \psi(z))}_{= \varphi}$

Hence, $\Phi \vDash \psi(\overline{\langle \varphi \rangle}) \to \varphi$ ■

Theorem 55. (Tarski)

Let $\Phi \subseteq FO(\tau_{ar})$ be consistent and all comp. functions be expressible in Φ.

If $\Phi^{\models}$ is expressible in Φ then $\Phi^{\models}$ is not complete.

Proof: Let $\Phi^{\models}$ be expressible in Φ via η, i.e.,

$$\psi \in \Phi^{\models} \quad \Rightarrow \quad \eta(\overline{\langle\psi\rangle}) \in \Phi^{\models}$$

$$\psi \notin \Phi^{\models} \quad \Rightarrow \quad \neg\eta(\overline{\langle\psi\rangle}) \in \Phi^{\models}$$

Since Φ is consistent, $\eta(\overline{\langle\psi\rangle}) \in \Phi^{\models} \Leftrightarrow \neg\eta(\overline{\langle\psi\rangle}) \notin \Phi^{\models}$

$\Rightarrow \psi \in \Phi^{\models}$

Hence, $\psi \in \Phi^{\models} \Leftrightarrow \eta(\overline{\langle\psi\rangle}) \in \Phi^{\models}$ (*)

By Thm. 54, there ex. φ s.t. $\Phi \models (\varphi \leftrightarrow \neg\eta(\overline{\langle\varphi\rangle}))$

We obtain

$$\begin{aligned}
\Phi \models \neg\varphi \quad &\Leftrightarrow \quad [\![\varphi]\!]^{\mathcal{I}} = 0 \text{ for each model } \mathcal{I} \text{ of } \Phi \\
&\Leftrightarrow \quad [\![\eta(\overline{\langle\varphi\rangle})]\!]^{\mathcal{I}} = 1 \text{ for each model } \mathcal{I} \text{ of } \Phi \\
&\Leftrightarrow \quad \Phi \models \eta(\overline{\langle\varphi\rangle}) \qquad (**)
\end{aligned}$$

Hence, $\varphi \in \Phi^{\models} \overset{(*)}{\Leftrightarrow} \eta(\overline{\langle\varphi\rangle}) \in \Phi^{\models} \overset{(**)}{\Leftrightarrow} \neg\varphi \in \Phi^{\models}$

Since Φ consistent, $\varphi, \neg\varphi \notin \Phi^{\models}$ ∎

Theorem 56. (Gödel)

Let $\Phi \subseteq FO(\tau_{Ar})$ be consistent ~~all~~ and all comp. functions be expressible in Φ.
If Φ is decidable then $\Phi^{\vdash}$ is incomplete.

Proof: Let Φ be decidable. Then, $\Phi^{\vdash}$ is enumerable.
Assume $\Phi^{\vdash}$ is complete. Then, $\Phi^{\vdash}$ is decidable.
So, $\Phi^{\vdash}$ is expressible in Φ. By Thm. 55,
$\Phi^{\vdash} = \Phi^{\models}$ is incomplete. $\lightning$ $\blacksquare$

Corollary 57.

An axiomatizable theory s.t. all comp. functions are expressible is incomplete.

Corollary 58.

The elementary arithmetic $Th(\mathfrak{N})$ is not axiomatizable.

Remark: Theorem 56 holds true for each vocabulary containing a const. symbol c, function symbol g:
use: $\overline{0} =_{def} c$, $\overline{n+1} =_{def} g(\overline{n}, \dots, \overline{n})$