2. First-order logic

2.1 Vocabularies and structures

Definition 1.

A vocabulary τ is a set of function and relation symbols. Each symbol has a fixed finite arity.
A vocabulary is called relational iff it consists only of relation symbols; it is called functional (or algebraic) iff it consists only of function symbols. Function symbols of arity 0 are called constant symbols.

Example:

1. vocabulary of arithmetics: $\tau_{Ar} = \{0, 1, +\}$; 0, 1 are constant symbols; + binary function symbol
2. vocabulary of ordered arithmetic: $\tau_{Ar}^{<} = \{0, 1, +, <\}$; $<$ binary relation symbol
3. vocabulary of graphs: $\tau_G = \{E\}$; E binary relation symbol

Notations:

- $P, Q, R, \dots, P_i, \dots$ relation symbols
- $f, g, h, \dots, f_i, \dots$ function symbols
- $c, d, \dots, c_i, \dots$ constant symbols
- $\sigma, \tau, \dots$ vocabularies

Definition 2.

Let τ be a vocabulary.

A pair $\mathfrak{A} = (A, \alpha)$ is called τ-structure iff

(1.) A is a non-empty set (universe)

(2.) $\alpha(c) \in A$ for all constant symbols $c \in \tau$

(3.) $\alpha(f) : A^n \to A$ for all n-ary function symbols $f \in \tau$

(4.) $\alpha(R) \subseteq A^n$ for all n-ary relation symbols $R \in \tau$.

Examples:

1. sets: $\tau = \emptyset$, i.e., $\mathfrak{A} = (A, \emptyset)$

2. graphs: $\tau_G = \{E\}$; E binary relation symbol:
   τ_G-structure $G = (V, E_G)$ is a directed graph,
   $E^G \subseteq V \times V$.

3. partial orders: $\tau = \{<\}$, $<$ binary relation symbol:
   $\{<\}$-structure $(A, <)$ is a partial order iff
   it is irreflexive, transitive, antisymmetric.

4. arithmetics: $\tau_{Ar} = \{0, 1, +, \cdot\}$; ~~The~~ τ_{Ar}-structures:
   - $(\mathbb{N}, 0, 1, +, \cdot)$ standard arithmetics
   - $(\mathbb{Z}, 0, 1, +, \cdot)$ or any other ring
   - $(\mathbb{R}; 0, 1, +, \cdot)$ or any other field
   - $(\mathbb{N} \cup \{\infty\}, 0, 1, +, \cdot)$; $a + \infty = \infty + a = a \cdot \infty = \infty \cdot a = \infty$

Some definitions / results omitted