We obtain:

$$[\![\varphi]\!]^I = \min_{m \in \mathbb{N}} \operatorname{seq}(2 \mid m, \max_{n \in \mathbb{N}} n \cdot 2 = m)$$

$$= \min_{m \in \mathbb{N}} \operatorname{seq}\left(2 \mid m, \begin{cases} 1 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}\right)$$

$$= \min \{ \underbrace{\operatorname{seq}(1,1)}_{m \text{ even}}, \underbrace{\operatorname{seq}(0,0)}_{m \text{ odd}} \}$$

$$= \min \{1, 1\}$$

$$= 1$$

2.3 Models and proofs in first-order logic

- τ vocabulary
- $\varphi, \psi, \varphi_1, \ldots \in FO(\tau)$
- $\Phi, \Psi, \Phi_1, \ldots \subseteq FO(\tau)$
- $I, I', I_1, \ldots$ interpretations

Definition 10.

Let τ be a vocabulary, let I be a τ-interpretation.

(1.) I is said to be a model of $\varphi \in FO(\tau)$ if and only if $[\![\varphi]\!]^I = 1$.

I is said to be a model of $\Phi \subseteq FO(\tau)$ if and only if $[\![\varphi]\!]^I = 1$ for all $\varphi \in \Phi$ (in symbols: $[\![\Phi]\!]^I = 1$)

If φ is a theorem then we also say that $\mathfrak{A}$ is a model of φ iff $I = (\mathfrak{A}, \beta)$ is a model of φ.

(2.) Φ is said to entail $\varphi \Leftrightarrow_{def} [\![\Phi]\!]^{I} \in [\![\varphi]\!]^{I}$ for all I (in symbols: $\Phi \models \varphi$)

(3.) $\varphi \models \psi \Leftrightarrow \{\varphi\} \models \psi$

(4.) $\Phi^{\models} =_{def} \{\varphi \mid \Phi \models \varphi\}$

Proposition 11.

$\models$ is a closure operator on $\mathcal{P}(FO(\tau))$.

Definition 12.

Let $\varphi, \psi \in FO(\tau)$ for vocabulary τ.

(1.) φ is true (or tautology) $\Leftrightarrow_{def} [\![\varphi]\!]^{I} = 1$ for all I

(2.) φ is satisfiable $\Leftrightarrow_{def}$. φ possesses a model

(3.) $\varphi \equiv \psi \Leftrightarrow_{def} [\![\varphi]\!]^{I} = [\![\psi]\!]^{I}$ for all I

Proposition 13.

Let $\varphi, \psi \in FO(\tau)$.

(1.) φ true $\Leftrightarrow \emptyset \models \varphi$

(2.) $(\varphi \to \psi)$ true $\Leftrightarrow \varphi \models \psi$

(3.) $(\varphi \leftrightarrow \psi)$ true $\Leftrightarrow \varphi \models \psi$ and $\psi \models \varphi$

$\Leftrightarrow \varphi \equiv \psi$

(4.) φ true $\Leftrightarrow \forall x \varphi$ true

(5.) $\forall x \varphi \equiv \neg \exists x \neg \varphi$

$\exists x \varphi \equiv \neg \forall x \neg \varphi$

Definition 14.

Let τ be a vocabulary.

(1.) A set $\Phi \subseteq FO_0(\tau)$ of theorems is called a theory iff Φ is satisfiable and $\Phi = \Phi^{\models}$.

(2.) A theory Φ is said to be complete if and only if $\varphi \in \Phi$ or $\neg\varphi \in \Phi$ for all $\varphi \in FO_0(\tau)$

How to define a theory?

(1.) Theory of a structure:

Let $\mathfrak{A} = (A, \alpha)$ be a τ-structure. Define

$Th(\mathfrak{A}) =_{def} \{\varphi \mid \varphi \in FO_0(\tau), \mathfrak{A} \text{ is a model of } \varphi\}$

Proposition 15.

$Th(\mathfrak{A})$ is a complete theory.

Proof: $\mathfrak{A}$ is a model of $Th(\mathfrak{A})$ by definition.
To show $Th(\mathfrak{A})^{\models} = Th(\mathfrak{A})$: let $\varphi \in Th(\mathfrak{A})^{\models}$, i.e., each model of $Th(\mathfrak{A})$ is a model of φ, i.e., $\mathfrak{A}$ is a model of φ; so, $\varphi \in Th(\mathfrak{A})$.
To show $Th(\mathfrak{A})$ complete: suppose $\varphi \in FO_0(\tau)$, $\varphi \notin Th(\mathfrak{A})$ so, $\mathfrak{A}$ is not a model of φ, i.e., $[\![\varphi]\!]^{\mathfrak{A}} = 0$.
Then, $[\![\neg\varphi]\!]^{\mathfrak{A}} = 1$. Hence, $\neg\varphi \in Th(\mathfrak{A})$. ■

(2.) Axiomatic theory:

Proposition 16.

If $\Phi \subseteq FO_0(\tau)$ is satisfiable then $\Phi^{\models}$ is a theory.

Examples:

(1.) (elementary) group theory: $\tau_{gr} =_{def} \{+, 0\}$; $+$ binary function symbol, 0 constant symbol

$$\Phi_{gr} =_{def} \{ \underbrace{\forall x \forall y \forall z \, ((x+y)+z = x+(y+z))}_{\text{associativity}}, \quad (++xyz = +x+yz)$$

$$\underbrace{\forall x \exists y \, (x+y=0)}_{\text{inverse element}},$$

$$\underbrace{\forall x \, (x+0=x)}_{\text{neutral element}} \}$$

τ_{gr}-structure $\mathfrak{A} = (A, \alpha)$ is said to be a group iff $\mathfrak{A}$ is a model of Φ_{gr} (i.e., all theorems in $\Phi_{gr}^{\vdash}$ hold for $\mathfrak{A}$)

(2.) theory of equivalence relations: $\tau_{eq} = \{R\}$; R binary relation symbol

$$\Phi_{eq} =_{def} \{ \underbrace{\forall x \, R(x,x)}_{\text{reflexivity}},$$

$$\underbrace{\forall x \forall y \, (R(x,y) \to R(y,x))}_{\text{symmetry}},$$

$$\underbrace{\forall x \forall y \forall z \, ((R(x,y) \land R(y,z)) \to R(x,z))}_{\text{transitivity}} \}$$

τ_{eq}-structure $\mathfrak{A} = (A, \alpha)$ is said to be an equivalence relation iff $\mathfrak{A}$ is model of Φ_{eq}.

(3.) theory of counting: $\tau = \emptyset$ (i.e., α irrelevant in τ-structure)

- $\varphi_2^{\geq} =_{def} \exists x_1 \exists x_2 \, \neg x_1 = x_2$; A is a model of $\varphi_2^{\geq} \Leftrightarrow |A| \geq 2$
- $\varphi_3^{\geq} =_{def} \exists x_1 \exists x_2 \exists x_3 \, (\neg x_1 = x_2 \land \neg x_2 = x_3 \land \neg x_1 = x_3)$

- $\varphi_k^{\geq} =_{def} \exists x_1 \ldots \exists x_k \bigwedge_{i=1}^{k-1} \bigwedge_{j=i+1}^{k} \neg x_i = x_j$;

  A is a model of $\varphi_k^{\geq}$ iff $\|A\| \geq k$.

- $\{\varphi_k^{\geq}\}^{\models}$ theory of sets containing at least k elements

- $\{\varphi_1^{\geq}, \varphi_2^{\geq}, \varphi_3^{\geq}, \ldots\}^{\models}$ theory of infinite sets

Entailment $\Phi \models \varphi$ not effectively to check, so derivations needed; restrict $FO(\tau)$ to

- $\neg, \vee$ instead of $\neg, \wedge, \vee, \rightarrow, (\leftrightarrow)$
- $\exists$ instead of $\exists, \forall$

Let $\varphi \in FO(\tau)$, $t \in T(\tau)$ for some vocabulary τ.
Then, $\varphi(x/t)$ is defined to be that formula that is obtained from φ when x is replaced with t whereever x is free in φ

Definition 17.

Let τ be a vocabulary. Let $\Phi \subseteq FO(\tau)$ be a set of formulas. The set $\Phi^{\vdash}$ is inductively defined as follows:

(1.) base case:

$$\varphi \in \Phi \Rightarrow \varphi \in \Phi^{\vdash}$$

$$t \in T(\tau) \Rightarrow (t = t) \in \Phi^{\vdash}$$

(2.) inductive step:

- case distinction, CD:

$$\left.\begin{array}{l} \varphi \in (\Phi \cup \{\psi\})^{\vdash} \\ \varphi \in (\Phi \cup \{\neg\psi\})^{\vdash} \end{array}\right\} \Rightarrow \varphi \in \Phi^{\vdash}$$

- indirect proof, IP:

$$\left.\begin{array}{l} \psi \in (\Phi \cup \{\neg\varphi\})^{\vdash} \\ \neg\psi \in (\Phi \cup \{\neg\varphi\})^{\vdash} \end{array}\right\} \Rightarrow \varphi \in \Phi^{\vdash}$$

- disjunction introduction in the premise; DIP:

$$\left.\begin{array}{l} \varphi \in (\Phi \cup \{\psi\})^{\vdash} \\ \varphi \in (\Phi \cup \{\psi'\})^{\vdash} \end{array}\right\} \Rightarrow \varphi \in (\Phi \cup \{\psi \vee \psi'\})^{\vdash}$$

- disjunction introduction in the conclusion; DIC:

$$\varphi \in \Phi^{\vdash} \Rightarrow (\varphi \vee \psi), (\psi \vee \varphi) \in \Phi^{\vdash}$$

- introducing $\exists$ in the conclusion; $\exists C$:

$$\varphi(x/t) \in \Phi^{\vdash} \Rightarrow \exists x \varphi \in \Phi^{\vdash}$$

- introducing $\exists$ in the premise; $\exists P$:

$$\varphi \in (\Phi \cup \{\psi(x/y)\})^{\vdash}, \ y \text{ not free in } \varphi, \psi, \Phi$$
$$\Rightarrow \varphi \in (\Phi \cup \{\exists x \psi\})^{\vdash}$$

- substitution; SUB:

$$\varphi(x/t) \in \Phi^{\vdash} \Rightarrow \varphi(x/t') \in (\Phi \cup \{t = t'\})^{\vdash}$$

- monotonicity; MO:

$$\varphi \in \Phi^{\vdash}, \ \Phi \subseteq \Psi \Rightarrow \varphi \in \Psi^{\vdash}$$

Finally, $\Phi \vdash \varphi$ denotes $\varphi \in \Phi^{\vdash}$.

Proposition 18.

$\vdash$ is a closure operator on $\mathcal{P}(FO(\tau))$.

Theorem 19. (Correctness theorem for FO)

$\Phi^{\vdash} \subseteq \Phi^{\models}$ for all sets $\Phi \subseteq FO(\tau)$.

Proof: (By induction on the structure of $\Phi^{\vdash}$)

- base case:

  $\varphi \in \Phi^{\vdash}$ because $\varphi \in \Phi$: Then, $\varphi \in \Phi^{\models}$

  $\varphi = (t=t) \in \Phi^{\vdash}$ because $t \in T(\tau)$: let $\mathcal{I}$ be a τ-interpretation s.t. $[\![\Phi]\!]^{\mathcal{I}} = 1$. We have $[\![t]\!]^{\mathcal{I}} = [\![t]\!]^{\mathcal{I}}$, so $[\![t=t]\!]^{\mathcal{I}} = 1$, thus, $(t=t) \in \Phi^{\models}$

- inductive step:

  (CD) $\varphi \in \Phi^{\vdash}$ because $\varphi \in (\Phi \cup \{\psi\})^{\vdash}$, $\varphi \in (\Phi \cup \{\neg\psi\})^{\vdash}$:
  By i.a., $\varphi \in (\Phi \cup \{\psi\})^{\models}$, $\varphi \in (\Phi \cup \{\neg\psi\})^{\models}$.
  Let $\mathcal{I}$ be a τ-interpretation s.t. $[\![\Phi]\!]^{\mathcal{I}} = 1$.
  Two cases (!):
  - $[\![\psi]\!]^{\mathcal{I}} = 1$: Then, $[\![\Phi \cup \{\psi\}]\!]^{\mathcal{I}} = 1$; so, $[\![\varphi]\!]^{\mathcal{I}} = 1$
  - $[\![\psi]\!]^{\mathcal{I}} = 0$: Then, $[\![\Phi \cup \{\neg\psi\}]\!]^{\mathcal{I}} = 1$; so, $[\![\varphi]\!]^{\mathcal{I}} = 1$

  Hence, $\varphi \in \Phi^{\models}$

  (CP) $\varphi \in \Phi^{\vdash}$ because $\psi, \neg\psi \in (\Phi \cup \{\neg\varphi\})^{\vdash}$:
  By i.a., $\psi, \neg\psi \in (\Phi \cup \{\neg\varphi\})^{\models}$
  Let $\mathcal{I}$ be a τ-interpretation s.t. $[\![\Phi]\!]^{\mathcal{I}} = 1$.
  Assume to the contrary (!), $[\![\varphi]\!]^{\mathcal{I}} = 0$. Then, $[\![\neg\varphi]\!]^{\mathcal{I}} = 1$.

Thus, $[\![\Phi \cup \{\neg\varphi\}]\!]^I = 1$; so, $[\![\psi]\!]^I = [\![\neg\psi]\!]^I = 1$. ↯

Hence, $[\![\varphi]\!]^I = 1$, i.e., $\varphi \in \Phi^{\models}$.

(DIC), (DIP): similar to (CD), (IP)

($\exists$C): $\exists x\varphi \in \Phi^{\vdash}$ because $\varphi(x|t) \in \Phi^{\vdash}$:

By i.a., $\varphi(x|t) \in \Phi^{\models}$.

Let I be a τ-interpretation s.t. $[\![\Phi]\!]^I = 1$.

Then, $[\![\varphi(x|t)]\!]^I = 1$. Define τ-interpretation $\hat{I}$:

$$\hat{I}(z) =_{def} \begin{cases} [\![z]\!]^I & \text{if } z \neq x \\ [\![t]\!]^I & \text{if } z = x \end{cases}$$

So, $\hat{I} \overset{x}{=} I$, $[\![\varphi]\!]^{\hat{I}} = [\![\varphi(x|t)]\!]^I$

Thus, $[\![\exists x\varphi]\!]^I = \max\limits_{I' \overset{x}{=} I} [\![\varphi]\!]^{I'}$

$$\geq [\![\varphi]\!]^{\hat{I}} = [\![\varphi(x|t)]\!]^I = 1$$

($\exists$D): $\varphi \in (\Phi \cup \{\exists x\psi\})^{\vdash}$ because $\varphi \in (\Phi \cup \{\psi(x|y)\})^{\vdash}$, y not free in φ, ψ, Φ: By i.a., $\varphi \in (\Phi \cup \{\psi(x|y)\})^{\models}$.

Let I be a τ-interpretation s.t. $[\![\Phi \cup \{\exists x\psi\}]\!]^I = 1$, i.e., there ex. τ-interpretation I' s.t. $I' \overset{x}{=} I$, $[\![\psi]\!]^{I'} = 1$. Define τ-interpretation I'':

$$I''(z) =_{def} \begin{cases} I(z) & \text{if } z \neq y \\ I'(x) & \text{if } z = y \end{cases}$$

So, $I'' \overset{y}{=} I$, $I''(y) = I'(x)$.

Cases:

(i) y not free in ψ: $[\![\psi(x|y)]\!]^{I''} = [\![\psi]\!]^{I'} = 1$

(ii) y not free in Φ: $[\![\Phi]\!]^{I''} = [\![\Phi]\!]^{I} = 1$

(iii) y not free in φ: $[\![\varphi]\!]^{I} = [\![\varphi]\!]^{I''} = 1$

Hence, $\varphi \in (\Phi \cup \{\exists x\psi\})^{\models}$.

(SGB), (MO): Exercise. ■

Theorem 20. (compactness theorem for FO)

Let $\Phi \subseteq FO(\tau)$, $\varphi \in FO(\tau)$. Then, $\varphi \in \Phi^{\vdash}$ if and only if $\varphi \in \Phi_0^{\vdash}$ for some finite $\Phi_0 \subseteq \Phi$.

Proof: Similar to compactness theorem for PL. ∎

Lemma 21.

The following rules are correct:

(1.) inconsistency; IC:

$$\psi, \neg\psi \in \Phi^{\vdash} \Rightarrow \varphi \in \Phi^{\vdash} \quad (\text{i.e., } \Phi^{\vdash} = FO(\tau))$$

(2.) chain rule; CH:

$$\psi \in \Phi^{\vdash},\ \varphi \in (\Phi \cup \{\psi\})^{\vdash} \Rightarrow \varphi \in \Phi^{\vdash}$$

(3.) modus ponens; MP:

$$\psi \in \Phi^{\vdash},\ \neg\psi \vee \varphi \in \Phi^{\vdash} \Rightarrow \varphi \in \Phi^{\vdash}$$

$$\neg\psi \in \Phi^{\vdash},\ \psi \vee \varphi \in \Phi^{\vdash} \Rightarrow \varphi \in \Phi^{\vdash}$$

Proof:

(1.)
$$\left.\begin{array}{l} \psi \in \Phi^{\vdash} \overset{\text{mo}}{\subseteq} (\Phi \cup \{\neg\varphi\})^{\vdash} \\ \neg\psi \in \Phi^{\vdash} \overset{\text{mo}}{\subseteq} (\Phi \cup \{\neg\varphi\})^{\vdash} \end{array}\right\} \overset{\text{IP}}{\Longrightarrow} \varphi \in \Phi^{\vdash}$$

(2.)
$$\left.\begin{array}{l} \psi \in \Phi^{\vdash} \overset{\text{mo}}{\subseteq} (\Phi \cup \{\neg\psi\})^{\vdash} \\ \neg\psi \in (\Phi \cup \{\neg\psi\})^{\vdash} \end{array}\right\} \overset{\text{IC}}{\Longrightarrow} \left.\begin{array}{l} \varphi \in (\Phi \cup \{\neg\psi\})^{\vdash} \\ \varphi \in (\Phi \cup \{\psi\})^{\vdash} \end{array}\right\} \overset{\text{CD}}{\Longrightarrow} \varphi \in \Phi^{\vdash}$$

(3.)
$$\left.\begin{array}{l} \neg\psi \in (\Phi \cup \{\neg\psi\})^{\vdash} \\ \psi \in (\Phi \cup \{\neg\psi\})^{\vdash} \end{array}\right\} \overset{\text{IC}}{\Longrightarrow} \left.\begin{array}{l} \varphi \in (\Phi \cup \{\neg\psi\})^{\vdash} \\ \varphi \in (\Phi \cup \{\varphi\})^{\vdash} \end{array}\right\} \overset{\text{DIP}}{\Longrightarrow} \underbrace{\varphi \in (\Phi \cup \{\neg\psi \vee \varphi\})^{\vdash}}_{(*)}$$

$$\left.\begin{array}{l} (*) \\ \neg\psi \vee \varphi \in \Phi^{\vdash} \end{array}\right\} \overset{\text{CH}}{\Longrightarrow} \varphi \in \Phi^{\vdash}$$

∎

Definition 22.

A set $\Phi \subseteq FO(\tau)$ of τ-formulas is said to be consistent if and only if there is no $\varphi \in FO(\tau)$ s.t. $\varphi, \neg\varphi \in \Phi^{\vdash}$.

Lemma 23.

Let $\Phi \subseteq FO(\tau)$ for some vocabulary τ.

(1.) Φ is satisfiable $\Rightarrow$ Φ is consistent

(2.) Φ is not consistent $\Leftrightarrow$ $\Phi^{\vdash} = FO(\tau)$.

Proof:

(1.) Assume Φ is not consistent, i.e., there is φ s.t. $\varphi, \neg\varphi \in \Phi^{\vdash}$. By Thm. 19., $\varphi, \neg\varphi \in \Phi^{\models}$. Then, $[\![\Phi]\!]^{I} \leq \min\{[\![\varphi]\!]^{I}, [\![\neg\varphi]\!]^{I}\} = 0$ for each model I. Hence, Φ is not satisfiable.

(2.) $\boxed{\Rightarrow}$ Φ is not consistent. Then, $\varphi, \neg\varphi \in \Phi^{\vdash}$. Hence, $FO(\tau) = \Phi^{\vdash}$ (1c).

$\boxed{\Leftarrow}$ $\Phi^{\vdash} = FO(\tau)$. Then, $\varphi, \neg\varphi \in \Phi^{\vdash}$ for all $\varphi \in FO(\tau)$. Hence, Φ is not consistent. ∎

Lemma 24.

$\Phi \subseteq FO(\tau)$ consistent $\Leftrightarrow$ each finite $\Phi_0 \subseteq \Phi$ consistent

Proof:

Φ not consistent

$\Leftrightarrow$ there ex. φ s.t. $\varphi, \neg\varphi \in \Phi^{\vdash}$

$\Leftrightarrow$ there ex. φ and finite $\Phi_0 \subseteq \Phi$ s.t. $\varphi, \neg\varphi \in \Phi_0^{\vdash}$

$\Leftrightarrow$ there ex. finite $\Phi_0 \subseteq \Phi$ not consistent ∎

Lemma 25.

Let $\Phi \subseteq FO(\tau)$ for some vocabulary τ, $\varphi \in FO(\tau)$.

(1.) $\varphi \notin \Phi^{\vdash} \Rightarrow \Phi \cup \{\neg\varphi\}$ consistent

(2.) $\varphi \in \Phi^{\vdash}$, Φ consistent $\Rightarrow \Phi \cup \{\varphi\}$ consistent

(3.) Φ consistent $\Leftrightarrow \Phi \cup \{\varphi\}$ consistent or $\Phi \cup \{\neg\varphi\}$ consistent

Proof: Exercise. ∎

Theorem 26.

Let τ be a vocabulary. Each consistent set $\Phi \subseteq FO(\tau)$ is satisfiable.

Proof: Later!

Theorem 27. (completeness theorem for FO; Gödel 1928)

Let τ be a vocabulary. Then, $\Phi^{\models} = \Phi^{\vdash}$ for each set $\Phi \subseteq FO(\tau)$.

Proof:

$\boxed{\supseteq}$ Thm. 19 (correctness)

$\boxed{\subseteq}$ Let $\varphi \in FO(\tau)$ be a τ-formula. Then,

$$\begin{aligned}
\varphi \notin \Phi^{\vdash} &\Rightarrow \Phi \cup \{\neg\varphi\} \text{ consistent} \\
&\overset{\text{Thm. 26}}{\Rightarrow} \Phi \cup \{\neg\varphi\} \text{ has a model } \mathcal{I} \\
&\Rightarrow [\![\Phi]\!]^{\mathcal{I}} = 1 \text{ and } [\![\neg\varphi]\!]^{\mathcal{I}} = 1 \\
&\Rightarrow [\![\Phi]\!]^{\mathcal{I}} = 1 \text{ and } [\![\varphi]\!]^{\mathcal{I}} = 0 \\
&\Rightarrow \varphi \notin \Phi^{\models}
\end{aligned}$$

∎

Corollary 28.

φ is true $\iff$ $\varphi \in \Phi^{\vdash}$

Recall: Φ is complete iff $\varphi \in \Phi$ or $\neg\varphi \in \Phi$ for all $\varphi \in FO(\tau)$

Lemma 29.

Let Φ be consistent and complete.

(1.) $\Phi = \Phi^{\vdash}$

(2.) $\neg\varphi \in \Phi \iff \varphi \notin \Phi$

(3.) $(\varphi \vee \psi) \in \Phi \iff \varphi \in \Phi$ or $\psi \in \Phi$ ∎

~~Define 30.~~

Definition 30.

Let τ be a vocabulary.

$\Phi \subseteq FO(\tau)$ is said to have witnesses if and only if for each $\varphi \in FO(\tau)$ and each variable x there is a τ-term $t \in T(\tau)$ s.t. $(\exists x \varphi \to \varphi(x/t)) \in \Phi^{\vdash}$.

Lemma 31.

Let Φ be consistent, complete, have witnesses. Then,

$\exists x \varphi \in \Phi \iff$ there is $t \in T(\tau)$ s.t. $\varphi(x/t) \in \Phi$

Proof:

$\boxed{\Rightarrow}$ Suppose $\exists x \varphi \in \Phi$. Since Φ has witnesses, L. 29 gives

$$\left.\begin{array}{r} (\exists x \varphi \to \varphi(x/t)) \in \Phi \\ \exists x \varphi \in \Phi \end{array}\right\} \xrightarrow{MP} \varphi(x/t) \in \Phi$$

$\boxed{\Leftarrow}$ Apply $\exists C$: $\varphi(x/t) \in \Phi \xrightarrow{\exists C} \exists x \varphi \in \Phi^{\vdash} = \Phi$ (by L. 29) ∎

Let $\Phi \subseteq FO(\tau)$. Define a relation $\sim_\Phi$ on $T(\tau)$:

$$t \sim_\Phi t' \;:\Longleftrightarrow_{def}\; (t=t') \in \Phi$$

<u>Lemma 32.</u>

Let Φ be consistent and complete.

(1.) $\sim_\Phi$ is an equivalence relation on $T(\tau)$

(2.) For each n-ary function symbol $f \in \tau$,

$t_1 \sim_\Phi t_1', \dots, t_n \sim_\Phi t_n' \Rightarrow f t_1 \dots t_n \sim_\Phi f t_1' \dots t_n'$

(3.) For each n-ary relation symbol $R \in \tau$,

$t_1 \sim_\Phi t_1', \dots, t_n \sim_\Phi t_n' \Rightarrow (R t_1 \dots t_n \in \Phi \Leftrightarrow R t_1' \dots t_n' \in \Phi)$

<u>Proof:</u>

(1.) $\sim_\Phi$ reflexive: $(t=t) \in \Phi^\vdash = \Phi \Rightarrow t \sim_\Phi t$

$\sim_\Phi$ symmetrical: let $t \sim_\Phi t'$, so $(t=t') \in \Phi$.

Define $\varphi(x) :=_{def} (x=t)$. So, $\varphi(x|t) = (t=t) \in \Phi$.

$\overset{(SUB)}{\Rightarrow} (t'=t) = \varphi(x|t') \in (\Phi \cup \{t=t'\})^\vdash = \Phi^\vdash = \Phi$

Thus, $t' \sim_\Phi t$

$\sim_\Phi$ transitive: let $t \sim_\Phi t'$, $t' \sim_\Phi t''$; so, $(t=t'), (t'=t'') \in \Phi$

Define $\varphi(x) :=_{def} (x=t'')$; so, $\varphi(x|t') = (t'=t'') \in \Phi$

$$\left.\begin{aligned} &\overset{(SUB)}{\Rightarrow} (t=t'') = \varphi(x|t) \in (\Phi \cup \{t'=t\})^\vdash \\ &t \sim_\Phi t' \overset{symm.}{\Rightarrow} t' \sim_\Phi t \Rightarrow (t'=t) \in \Phi \end{aligned}\right\} \Rightarrow (t=t'') \in \Phi$$

Hence, $t \sim_\Phi t''$

(2), (3) straightforward. ■

Proof plan:

- Want to show: Φ consistent $\Rightarrow$ Φ has a model
- Show first: Φ consistent, complete, has witnesses $\Rightarrow$ Φ has a model
- show next: Φ consistent $\Rightarrow$ there is $\Psi \supseteq \Phi$ s.t. Ψ consistent, complete, has witnesses

Define a model I_Φ for $\Phi \subseteq FO(\tau)$; $I_\Phi = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (A, \alpha)$

- A is set of equivalence classes of $\sim_\Phi$
- t^* denotes equivalence class of $t \in T(\tau)$
- c constant symbol $\Rightarrow$ $\alpha(c) =_{def} c^*$
- x variable $\Rightarrow$ $\beta(x) =_{def} x^*$
- f function symbol $\Rightarrow$ $\alpha(f) =_{def} f^*$ where $f^*(t_1^*, \dots, t_n^*) =_{def} (f t_1 \dots t_n)^*$
- R relation symbol $\Rightarrow$ $\alpha(R) =_{def} R^*$ where $(R^*(t_1^*, \dots, t_n^*) = 1 \iff_{def} R t_1 \dots t_n \in \Phi)$

Model is well-defined (by Lemma 32) for consistent, complete Φ

Lemma 33.

Let $\Phi \subseteq FO(\tau)$ be consistent, complete. Let $t \in T(\tau)$.
Then, $[\![t]\!]^{I_\Phi} = t^*$.

Proof: (induction on term structure)

- base case: c constant symbol $\Rightarrow [\![c]\!]^{I_\Phi} = \alpha(c) = c^*$
  x variable $\Rightarrow [\![x]\!]^{I_\Phi} = \beta(x) = x^*$
- inductive step: f function symbol, $t_1, \dots, t_n \in T(\tau)$
  $\Rightarrow [\![f t_1 \dots t_n]\!]^{I_\Phi} = \alpha(f)([\![t_1]\!]^{I_\Phi}, \dots, [\![t_n]\!]^{I_\Phi})$

$\overset{\text{i.a.}}{\Rightarrow} f^*(t_1^*, \dots, t_n^*) = (f t_1 \dots t_n)^*$ ∎

Theorem 34. (Henkin)

Let $\Phi \subseteq FO(\tau)$ be consistent, complete, have witnesses.
For each $\varphi \in FO(\tau)$,

$$[\![\varphi]\!]^{I_\Phi} = 1 \iff \varphi \in \Phi$$

Proof: (induction on formula structure)

- base case:
  - $$[\![t = t']\!]^{I_\Phi} = 1 \iff [\![t]\!]^{I_\Phi} = [\![t']\!]^{I_\Phi} \overset{L.33}{\iff} t^* = t'^* \iff t \sim_\Phi t' \iff (t = t') \in \Phi$$
  - $$[\![R t_1 \dots t_n]\!]^{I_\Phi} = 1 \iff \alpha(R)([\![t_1]\!]^{I_\Phi}, \dots, [\![t_n]\!]^{I_\Phi}) = 1 \overset{L.33}{\iff} R^*(t_1^*, \dots, t_n^*) = 1 \iff R t_1 \dots t_n \in \Phi$$

- inductive step:
  - $$[\![\neg\varphi]\!]^{I_\Phi} = 1 \iff [\![\varphi]\!]^{I_\Phi} = 0 \overset{\text{i.a.}}{\iff} \varphi \notin \Phi \overset{\Phi \text{ compl. cons.}}{\iff} \neg\varphi \in \Phi$$
  - $$[\![\varphi \vee \psi]\!]^{I_\Phi} = 1 \iff [\![\varphi]\!]^{I_\Phi} = 1 \text{ or } [\![\psi]\!]^{I_\Phi} = 1 \overset{\text{i.a.}}{\iff} \varphi \in \Phi \text{ or } \psi \in \Phi \overset{L.29}{\iff} (\varphi \vee \psi) \in \Phi$$
  - $\exists x \varphi \in \Phi \overset{L.31}{\Rightarrow}$ there is $t \in T(\tau)$ s.t. $\varphi(x/t) \in \Phi$
    $\overset{\text{i.a.}}{\Rightarrow}$ there is $t \in T(\tau)$ s.t. $[\![\varphi(x/t)]\!]^{I_\Phi} = 1$

    Define
    $$I'(z) =_{def} \begin{cases} I_\Phi(z) & \text{if } z \neq x \\ I_\Phi(t) \ (= [\![t]\!]^{I_\Phi}) & \text{if } z = x \end{cases}$$

Then, $\mathcal{I}' \overset{x}{=} \mathcal{I}_\Phi$, $[\![\varphi(x/t)]\!]^{\mathcal{I}_\Phi} = [\![\varphi]\!]^{\mathcal{I}'} = 1$

Thus, $[\![\exists x \varphi]\!]^{\mathcal{I}_\Phi} = \max\limits_{\mathcal{I} \overset{x}{=} \mathcal{I}_\Phi} [\![\varphi]\!]^{\mathcal{I}} \geq [\![\varphi]\!]^{\mathcal{I}'} = 1$

Now, $[\![\exists x \varphi]\!]^{\mathcal{I}_\Phi} = 1$. Then, $[\![\varphi]\!]^{\mathcal{I}} = 1$ for $\mathcal{I} \overset{x}{=} \mathcal{I}_\Phi$.

Let t be τ-term s.t. $\mathcal{I}(x) = t^*$. Thus, $[\![\varphi(x/t)]\!]^{\mathcal{I}_\Phi} = [\![\varphi]\!]^{\mathcal{I}} = 1$

$\overset{\text{i.a.}}{\Rightarrow} \varphi(x/t) \in \Phi \overset{4.29}{\Rightarrow} \exists x \varphi \in \Phi$ ■

Corollary 35.

Let Φ be consistent, complete, have witnesses.
Then, Φ is satisfiable.

Define $\text{free}(\Phi) = \bigcup\limits_{\varphi \in \Phi} \text{free}(\varphi)$

Lemma 36.

Let τ be a vocabulary.

Let Φ be consistent s.t. $\text{free}(\Phi)$ is finite.

Then, there is a consistent $\Psi \subseteq FO(\tau)$ s.t. $\Psi \supseteq \Phi$ and Ψ has witnesses.

Proof: Suppose $\exists z_1 \varphi_1, \exists z_2 \varphi_2, \ldots$ are $\exists$-formulas (countable, since $\Sigma(\tau)$ countable). Define Φ_i as follows:

$\Phi_0 =_{def} \Phi$

for $n > 0$, choose $y_n \notin \text{free}(\Phi_{n-1} \cup \{\psi_{n-1}\})$

define $\psi_n =_{def} (\exists z_n \varphi_n \rightarrow \varphi_n(z_n/y_n))$

$\Phi_n =_{def} \Phi_{n-1} \cup \{\psi_n\}$

Corollary 37.

Let τ be a vocabulary.
Let $\underline{\Phi} \subseteq FO(\tau)$ be consistent and $free(\underline{\Phi})$ is finite. Then, $\underline{\Phi}$ has a model.

Theorem 26.

Let τ be a vocabulary.
Each consistent set $\underline{\Phi} \subseteq FO(\tau)$ is satisfiable.

Proof: Let $c_0, c_1, c_2, \dots$ be new constant symbols, $c_i \notin \tau$. Consider $\tau' =_{def} \tau \cup \{c_0, c_1, c_2, \dots\}$.
Let $\varphi \in FO(\tau)$, $free(\varphi) = \{x_{i_1}, \dots, x_{i_n}\}$. Define
$\varphi' =_{def} \varphi(x_{i_1}/c_{i_1}, \dots, x_{i_n}/c_{i_n}) \in FO(\tau')$.
That is, $free(\varphi') = \emptyset$.
Define $\underline{\Phi}' =_{def} \{\varphi' \mid \varphi \in \underline{\Phi}\} \subseteq FO(\tau')$.
So, $free(\underline{\Phi}') = \emptyset$.
It holds that $\underline{\Phi}'$ is consistent (Δ).
By Cor. 37, $\underline{\Phi}'$ has a model $\mathcal{I}' = (\mathfrak{A}', \beta)$, $\mathfrak{A}' = (A, \alpha')$.
Define τ- interpretation $\mathcal{I} = (\mathfrak{A}, \beta)$, $\mathfrak{A} = (A, \alpha)$:

- $\alpha(f) =_{def} \alpha'(f)$ for $f \in \tau$
- $\alpha(R) =_{def} \alpha'(R)$ for $R \in \tau$
- $\alpha(c) =_{def} \alpha'(c)$ for $c \in \tau$
- $\beta(x_k) =_{def} \alpha'(c_k)$ for $k \in \mathbb{N}$

$\Psi =_{def} \bigcup_{n\geq 0} \Phi_n$

Obviously, $\Phi \subseteq \Psi$ and Ψ has witnesses.

We have to show: Ψ is consistent;

sit is enough to show: Φ_n is consistent

Induction on n:

- $n=0$: $\Phi_0 = \Phi$ consistent
- $n>0$: By i.a., Φ_{n-1} is consistent

Assume $\Phi_n = \Phi_{n-1} \cup \{\varphi_n\}$ not consistent.

Then, $\Phi_n^{\vdash} = FO(\tau)$. Choose $\eta \in \Phi_{n-1}$:

$\neg\eta \in (\Phi_{n-1} \cup \{\varphi_n\})^{\vdash} = (\Phi_{n-1} \cup \{\neg\exists z_n \varphi_n \vee \varphi_n(z_n/y_n)\})^{\vdash}$

$\neg\exists z_n \varphi_n \in (\Phi_{n-1} \cup \{\neg\exists z_n \varphi_n\})$

$\overset{DIP}{\Rightarrow} (\neg\exists z_n \varphi_n \vee \varphi_n(z_n/y_n)) \in (\Phi_{n-1} \cup \{\neg\exists z_n \varphi_n\})^{\vdash}$

$\overset{CH}{\Rightarrow} \neg\eta \in (\Phi_{n-1} \cup \{\neg\exists z_n \varphi_n\})^{\vdash}$

$\varphi_n(z_n/y_n) \in (\Phi_{n-1} \cup \{\varphi_n(z_n/y_n)\})^{\vdash}$

$\overset{DIP}{\Rightarrow} (\neg\exists z_n \varphi_n \vee \varphi_n(z_n/y_n)) \in (\Phi_{n-1} \cup \{\varphi_n(z_n/y_n)\})^{\vdash}$

$\overset{CH}{\Rightarrow} \neg\eta \in (\Phi_{n-1} \cup \{\varphi_n(z_n/y_n)\})^{\vdash}$

$\overset{\exists P}{\Rightarrow} \neg\eta \in (\Phi_{n-1} \cup \{\exists z_n \varphi_n\})^{\vdash}$, y_n not free in $\Phi_{n-1}, \eta, \neg\eta$.

Thus, by CD: $\neg\eta \in \Phi_{n-1}^{\vdash}$

Hence, $\eta, \neg\eta \in \Phi_{n-1}^{\vdash}$, i.e., Φ_{n-1} not consistent $\lightning$

Hence, $[\![\varphi]\!]^{\mathcal{I}} = [\![\varphi']\!]^{\mathcal{I}'}$ for all $\varphi \in FO(\tau)$

Thus, $[\![\Phi]\!]^{\mathcal{I}} = [\![\Phi']\!]^{\mathcal{I}'} = 1$; so, Φ has a model. $\blacksquare$