Theorem 7.

$SO(\tau)$ is incomplete for each vocabulary τ, i.e., there ex. no set of derivation rules which is correct and complete.

Proof: Suppose there is a set of derivation rules such that $\vdash$ is defined which is correct, i.e., $\Phi^{\vdash} \subseteq \Phi^{\models}$, and complete, i.e., $\Phi^{\models} \subseteq \Phi^{\vdash}$.
Thus, $\Phi^{\models} = \Phi^{\vdash}$ for all sets $\Phi \subseteq SO(\tau)$

However, applying a derivation rules connects a finite number of formulas to obtain a new formula. Thus, compactness theorem always holds for $\vdash$. Since $\Phi^{\models} = \Phi^{\vdash}$, by assumption, the compactness theorem holds for $\models$. $\lightning$ ∎

4. Finite model theory

want to prove that $(aa)^+$ is not FO-definable

4.1 Predicate logic on words

Words as structures:

- Σ finite, non-empty alphabet
- vocabulary $\tau_\Sigma :=_{def} \{<\} \cup \{P_a \mid a \in \Sigma\}$ where
  $<$ is a binary relation symbol
  P_a is a unary relation symbol

- word $w \in \Sigma^*$ is identified with τ_Σ-structure $\mathfrak{A}_w = (A_w, \alpha_w)$:

$A_w =_{def} \{1, 2, \ldots, |w|\}$ positions in w

$\alpha_w(<) =_{def} \{(p,q) \mid p < q\}$ linear ordering of pos.

$\alpha_w(P_a) =_{def} \{p \mid w_p = a\}$.

4.1.1 First-order logic and star-free languages

Let φ be an $FO(\tau_\Sigma)$-sentence, i.e., $\varphi \in FO_0(\tau_\Sigma)$. Then,

$$L(\varphi) =_{def} \{w \in \Sigma^* \mid [\![\varphi]\!]^{\mathfrak{A}_w} = 1\}$$

is the language defined by φ.

Extend $L(\varphi)$ to arbitrary $FO(\tau_\Sigma)$-formulas:

- let $x_1, \ldots, x_\ell$ be all variables occurring in φ; w.l.o.g. bound variables are pairwise distinct and distinct to free variables
- let $(w, \bar{p}) = (w, (p_1, \ldots, p_\ell))$ denote τ_Σ-interpretation $(\mathfrak{A}_w, \beta)$ where $\beta(x_i) = p_i$ for $i \in \{1, \ldots, \ell\}$
- semantics of φ given $(w, (p_1, \ldots, p_\ell)) = (\mathfrak{A}_w, \beta)$: it is enough to specify for atomic formulas:

$$[\![x_i = x_j]\!]^{(w,\bar{p})} = 1 \iff_{def} p_i = p_j$$

$$[\![x_i < x_j]\!]^{(w,\bar{p})} = 1 \iff_{def} p_i < p_j$$

$$[\![P_a(x_i)]\!]^{(w,\bar{p})} = 1 \iff_{def} w_{p_i} = a$$

Example: $\varphi = \exists x\, P_a(x)$, i.e., $\mathrm{free}(\varphi) = \emptyset$, single bound var.

$$\llbracket \varphi \rrbracket^{(w,p)} = \max_{I' \overset{x}{=} (w,p)} \llbracket P_a(x) \rrbracket^{I'}$$

$$= \max_{p \in \{1,\dots,|w|\}} \llbracket P_a(x) \rrbracket^{(w,p)}$$

$$= \begin{cases} 1 & \text{if letter } a \text{ occurs in } w \\ 0 & \text{otherwise} \end{cases}$$

That is, $L(\varphi) = \Sigma^* a \Sigma^*$

Goal: $\varphi \in FO_0(\tau_\Sigma) \Rightarrow L(\varphi)$ regular

Need to encode assignments $\beta : \{x_1, \dots, x_k\} \to \{1, \dots, |w|\}$:

- let V be a set of variables
- $\Sigma_V =_{def} \Sigma \times \{0,1\}^V$ finite alphabet, $\{0,1\}^V = \{\sigma \mid \sigma : V \to \{0,1\}\}$
- encode (w, β) as

$$\overline{(w,\beta)} =_{def} (w_1, \sigma_1)(w_2, \sigma_2) \dots (w_n, \sigma_n) \text{ where}$$
$$\sigma_i(x_j) = 1 \iff \beta(x_j) = i$$

e.g., $\overline{(abaa, (1,2,1,3))} = \underline{a\,1010}\ \underline{b\,0100}\ \underline{a\,0001}\ \underline{a\,0000}$

$\overline{(abc, (2,2,2,3,3))} = \underline{a\,00000}\ \underline{b\,11100}\ \underline{c\,00011}$

- note: not all $v \in \Sigma_V^*$ correspond to (w, β), β assignment
  e.g. a1b1
- define $N_V \subseteq \Sigma_V^*$: $\overline{(w,\beta)} \in N_V \iff_{def} w \in \Sigma^*, \beta : V \to \{1, \dots, |w|\}$
- N_V is regular: $\Sigma_V^{x=1} =_{def} \{(a,\sigma) \mid \sigma(x) = 1\}$, $\Sigma_V^{x=0} =_{def} \{(a,\sigma) \mid \sigma(x) = 0\}$
  $\cancel{\text{e.g. } \Sigma_V^{y=1} = \{(b, 11100)(a,}$ $\Sigma_V^{x=1} \cup \Sigma_V^{x=0} = \Sigma_V$

Then,

$$N_V = \bigcap_{x \in V} (\Sigma_V^{x=0})^* \cdot \Sigma_V^{x=1} \cdot (\Sigma_V^{x=0})^*$$

- language $[\![\varphi]\!]_V$ of φ and V containing ~~a~~ free var. of φ:
  - atomic formulas:

$$[\![x=y]\!]_V =_{\text{def}} \{ \overline{(w,\beta)} \in N_V \mid \beta(x) = \beta(y) \}$$

$$[\![x<y]\!]_V =_{\text{def}} \{ \overline{(w,\beta)} \in N_V \mid \beta(x) < \beta(y) \}$$

$$[\![P_a(x)]\!]_V =_{\text{def}} \{ \overline{(w,\beta)} \in N_V \mid w_{\beta(x)} = a \}$$

  - composite formulas:

$$[\![\varphi \vee \psi]\!]_V =_{\text{def}} [\![\varphi]\!]_V \cup [\![\psi]\!]_V$$

$$[\![\neg\varphi]\!]_V = N_V \setminus [\![\varphi]\!]_V$$

$$[\![\exists x \varphi]\!]_V =_{\text{def}} \{ \overline{(w,\beta)} \in N_V \mid \text{there ex. } i \in \{1,\dots,|w|\} \text{ s.t. } \overline{(w,\beta[x \to i])} \in [\![\varphi]\!]_{V \cup \{x\}} \}$$

Proposition 3.

Let $\varphi \in FO(\tau_\Sigma)$, let V be the set of ~~all~~ free variables of φ. Then, $[\![\varphi]\!]_V$ is regular.

Proof: (Induction)

- base case: We obtain for atomic formulas:

$$[\![x=y]\!]_V = N_V \cap \Sigma_V^* \cdot (\Sigma_V^{x=1} \cap \Sigma_V^{y=1}) \cdot \Sigma_V^*$$

$$[\![x<y]\!]_V = N_V \cap \Sigma_V^* \cdot \Sigma_V^{x=1} \cdot \Sigma_V^* \cdot \Sigma_V^{y=1} \cdot \Sigma_V^*$$

$[\![P_a(x)]\!]_V = N_V \cap \Sigma_V^* \cdot \{(a,\sigma) \mid \sigma(x)=1\} \cdot \Sigma_V^*$

- inductive step:

$\varphi = \psi \vee \eta$: suppose $[\![\psi]\!]_V$, $[\![\eta]\!]_V$ regular,

then $[\![\varphi]\!]_V = [\![\psi]\!]_V \cup [\![\eta]\!]_V$ regular

$\varphi = \neg\psi$: suppose $[\![\psi]\!]_V$ is regular; then,

$[\![\varphi]\!]_V = N_V \setminus [\![\psi]\!]_V = N_V \cap \overline{[\![\psi]\!]_V}$ regular

$\varphi = \exists x \psi$: suppose $[\![\psi]\!]_{V\cup\{x\}}$ is regular

$[\![\psi]\!]_{V\cup\{x\}} = \Sigma_{V\cup\{x\}}^* \cdot \Sigma_{V\cup\{x\}}^{x=1} \cdot \Sigma_{V\cup\{x\}}^* \cap [\![\psi]\!]_{V\cup\{x\}}$ regular

It follows that $[\![\varphi]\!]_V$ is regular since reg. languages are closed under projections ∎

Corollary 4.

If $\varphi \in FO_0(\tau_\Sigma)$ then $L(\varphi)$ is regular.

Definition 5.

Let Σ be a finite alphabet.
The class $SF(\Sigma)$ of star-free languages is defined as follows:

(1.) base case:
$\{a\}$ is star-free for all $a \in \Sigma$
Σ^* is star-free

(2.) inductive step: if A, B are star-free then

$A \cup B$ is star-free
$A \cdot B$ is star-free
$\Sigma^* \setminus A = \overline{A}$ is star-free

Remarks:

- in star-free languages, iteration $(^*)$ is replaced w. $(^-)$
- $A \cap B = \overline{\overline{A} \cup \overline{B}}$ is star-free if A, B are star-free

Examples:

(1.) $\emptyset = \Sigma^* \setminus \Sigma^*$ is star-free

(2.) $\Delta \subseteq \Sigma$: Δ^* is star-free

$$\Delta^* = \Sigma^* \setminus \left(\Sigma^* \cdot (\Sigma \setminus \Delta) \cdot \Sigma^* \right) = \overline{\Sigma^* \cdot (\Sigma \setminus \Delta) \cdot \Sigma^*}$$

(3.) $(ab)^*$ is star-free ($\Sigma = \{a, b\}$), since

$$(ab)^* = \Sigma^* \setminus (\Sigma^* aa \Sigma^* \cup \Sigma^* bb \Sigma^* \cup \Sigma^* a \cup b \Sigma^*)$$

(4.) $(aa)^*$ is not star-free

Theorem 6. (Schützenberger 1965)
Let $L \in SF(\Sigma)$ be a star-free language. Then, for all $x, z, z' \in \Sigma^*$ there ex. n_0 s.t. for all $n \geq n_0$,

$$z x^n z' \in L \iff z x^{n+1} z' \in L$$

$\blacksquare$

Theorem 7. (McNaughton-Papert 1971; Pt. 1)
If $\varphi \in FO_0(\tau_\Sigma)$ then $L(\varphi)$ is star-free

Proof: (sketch) Reconstruct the proof of Prop. 3 using star-free languages:

- N_V is star-free
- $[\![x=y]\!]_V$ is star-free
- $[\![x<y]\!]_V$ is star-free
- $[\![P_a(x)]\!]_V$ is star-free
- $[\![\psi \vee \eta]\!]_V = [\![\psi]\!]_V \cup [\![\eta]\!]_V$ star-free
- $[\![\neg\psi]\!]_V = N_V \setminus [\![\psi]\!]_V = N_V \cap \overline{[\![\psi]\!]_V}$ is star-free
- $[\![\exists x \psi]\!]_V$ is star-free: $V' =_{def} V \cup \{x\}$

  $[\![\exists x \psi]\!]_{V'} = \Sigma_{V'}^{*} \cdot \Sigma_{V'}^{x=1} \cdot \Sigma_{V'}^{*} \cap N_{V'} \cap [\![\psi]\!]_{V'}$

  is star-free (by i.a.)

  special proj. (bijective renaming) $V' \to V$ maintains star-freeness of $[\![\exists x \psi]\!]_V$ (proof om.) ∎